Asymptotics of eigenvalue sums when some turning points are complex: Supplemental Information

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CONTENTS

1. Eigenvalues 1
2. Coefficients 2
3. Numerical values of D sums 18
4. Subdominant Corrections to Eigenvalues and Sums 19
5. Errors around the order of optimal truncation for various approximations 20

1. EIGENVALUES

j	ϵ_j
0	0.53018104524209144982352300834633177275760
1	1.89983651490069708439154709425628447888303
2	3.72784896899336919607829567359288374406891
3	5.82237275568908101042518664068546821827544
4	8.13091300942511296894747721519230676712229
5	10.61918645911797001207485555679431818835242
6	13.26423559184125909590691409184061396448786
7	16.04929885548416331713605321916594834377996
8	18.96150051351699257325818927595497722773812
9	21.99057904864486539265905687641368239738974
10	25.12812725834145951987229405263137996805062
11	28.36710702758651802365083551672587875609325
12	31.70152349335944748604674246147231780864901
13	35.12619731430829544456542596900956718356056
14	38.63660024099198285472718722061650648880103
15	42.22873313747097946444416863827981122051197
16	45.89903340449556703492441333126139251356587
17	49.64430333024663982290990336326178369064809
18	53.46165369086626282653755106808336737954231
19	57.34845869249258776907546415989160724528593

TABLE S1. The first 20 pure quartic oscillator eigenvalues to 41 decimal places.

2. COEFFICIENTS

n	a_n	$a_n^{(0)}/a_n$
1	$-8.5677483945836929757806241727690625614706 4 \times 10^{-02}$	1.21585
2	$2.1484375000000000000000000000000000000000 \times 10^{-02}$	1.58680
3	$5.2399367460103653525054155910802456837536 0 \times 10^{-02}$	1.27753
4	$-3.1885283333914620535714285714285714285714 3 \times 10^{-01}$	1.03061
5	$-2.9062535816646928843286846985051556177295 0 \times 10^{+00}$	1.03611
6	$4.1495818240060047669844193892045454545454 5 \times 10^{+01}$	1.06868
7	$9.1060511114531858275928496351554328409083 2 \times 10^{+02}$	1.05187
8	$-2.7591782332508906620205380022525787353515 6 \times 10^{+04}$	1.03384
9	$-1.0864978894694000134583657630919911752844 6 \times 10^{+06}$	1.03106
10	$5.4441988168756793641875714827524222384550 1 \times 10^{+07}$	1.03031
11	$3.3965135598086749457165482246053546709178 2 \times 10^{+09}$	1.02688
12	$-2.5755675623010452217153815427931224709802 7 \times 10^{+11}$	1.02374
13	$-2.3306660203713174721639306437454509227083 8 \times 10^{+13}$	1.02186
14	$2.4825929261622403315201136683860831136148 2 \times 10^{+15}$	1.02034
15	$3.0755866653177123679139406945634489407035 7 \times 10^{+17}$	1.01886
16	$-4.3840189490453944852985480175546439002240 0 \times 10^{+19}$	1.01755
17	$-7.1239318408738829585405318966213708534942 4 \times 10^{+21}$	1.01646
18	$1.3091564626310175285690706065712789499477 4 \times 10^{+24}$	1.01550
19	$2.7014739300863957878793892300666718152186 6 \times 10^{+26}$	1.01463
20	$-6.2199435138291640527951023883500568067905 4 \times 10^{+28}$	1.01385
21	$-1.5888283383807296998512248024206282331546 8 \times 10^{+31}$	1.01316
22	$4.4796870452834936824994783702975309474650 3 \times 10^{+33}$	1.01253
23	$1.3876574431153873213003736807736395775201 5 \times 10^{+36}$	1.01196
24	$-4.7026735606790001265034234933842335449106 7 \times 10^{+38}$	1.01144
25	$-1.7368304931458955174775635436457278668225 2 \times 10^{+41}$	1.01096
26	$6.9659543785712346948964180509177429902763 9 \times 10^{+43}$	1.01052
27	$3.0241042975617892439008232350253281853211 7 \times 10^{+46}$	1.01011
28	$-1.4167594390285636817398336939121884612131 6 \times 10^{+49}$	1.00974

TABLE S2. Numerical values of the a_n, which give the implicit WKB series in Eq. (3), to 41 digits. The last column compares the exact a_n values and their asymptotic approximation in Eq. (4).

n	p_n	l_n	k_n
1	2	1	-3
2	9	1	11
3	11	5	4697
4	19	7	-1170195
5	21	1	-53352893
6	28	11	122528437805
7	30	13	111267380793177
8	39	1	-15168742752828973
9	41	17	-3555512335821253125455
10	48	19	291157089685769931700559
11	50	1	3347554798932775282047461l
12	58	69	-5122261970274624999190381273637
13	60	25	-5880501324247971990184665597563193
14	67	3	1099098152354797471067971156751945725
15	69	29	46088250967707030420298436581845312064569
16	79	31	-821492823683019155454173965639590251580672675
17	81	1	-150780077967480130755832616092731868698927522489
18	88	1	405164300695037828279520928163036387727976768116105
19	90	37	108317012648026942903386914638255410623464583259298380435
20	98	1	-19711787821728055236321686070825368709206985572900140845915
21	100	41	-7228612752096796108130619005742789281887769084469843939816343931
22	107	43	31255443556722496241668245986603096470086980248323628641592233691755
23	109	5	39419976434342742317008441255340791396603989056753318185174343002409111
24	118	47	-7344837288473172027052212004321320027998623445687561837682548693750403819975
25	120	7	-14146516619967537788099399224549329185689570015293499543540143860474909795571579
26	127	1	11851957219017473817390003253898160007504967904864927337956139967867712717420236025
27	129	53	9548532590293074145391628258280448686048147750089961351657036562840799168683740155978923
28	137	1	-2468343066983529738651133976858499791974673754482893416205160897916904488706522462241790585

TABLE S3. Exact values of $a_n = k_n s_n/(2^{p_n} l_n)$ where $s_n = 1$ for n even and $2\pi^2/\gamma^4$ for n odd.

n	m	h_{nm}	β_{nm}	$h_{n+1,m}$	$\beta_{n+1,m}$
2	0	1	5	0	−15
	1	1	3	1	−27
	2	0	0	1	−3
4	0	3	−1105	0	1695
	1	2	−663	1	5085
	2	3	−339	0	1026
	3	2	−3	1	183
6	0	4	414125	0	−472200
	1	4	745425	0	−991620
	2	4	411807	2	−2783655
	3	4	72003	3	−1482333
	4	1	279	1	−30213
	5	0	0	3	−1125
8	0	7	−1282031525	0	242183775
	1	5	−769218915	1	1307792385
	2	6	−1309784193	1	1309208985
	3	3	−59030685	2	1195302885
	4	7	−133259805	2	245632779
	5	5	−1270125	2	19025361
	6	5	−2259	2	322875
10	0	8	1683480621875	0	−198147676875
	1	8	5050441865625	1	−1307774667375
	2	7	2906215577115	0	−850525792155
	3	7	1608516428001	2	−2215233100245
	4	8	877578556635	3	−1514950290315
	5	8	107795124945	4	−521051651091
	6	6	1149165855	4	−39029897259
	7	6	7775973	4	−896364225
	8	0	0	4	−1947375
12	0	10	−6718940277925125	0	236869405180500
	1	9	−12094092500265225	0	923790680203950
	2	10	−35087514532179465	0	1478540642750820
	3	9	−13145122230291663	0	1253451684627285
	4	10	−10840741540025859	4	9699776263059975
	5	9	−1211960628282915	5	5372585515629225
	6	10	−270172953308475	2	101367611128359
	7	9	−6185820878865	4	29664178933599
	8	8	−36476078301	4	820844217225
	9	7	−7796277	5	9168548625
14	0	11	18962375127249928125	0	−389616942676537500
	1	11	79641975534449698125	0	−1753276242044418750
	2	11	139408731621746078625	0	−3331899937689160050
	3	11	131951468918137428585	0	−3483339073562046720
	4	11	73314413381448097227	2	−8751228555415953915
	5	11	24306644435312436723	3	−6774563883273647175
	6	11	4678881922950432435	6	−12766988178649565235
	7	11	485308177639627275	7	−3488001530534142813
	8	9	5837824849337145	0	−1958941075095954
	9	9	95185955417427	6	−3940950046985775
	10	5	13765136193	6	−36346512879225
	11	0	0	7	−55085194125
16	0	15	−575440151532675686278125	0	844097335215098919375
	1	12	−345264090919605411766875	1	8609792819194008977625
	2	13	−1415816009984447362360125	1	18944115674048484898275
	3	11	−405117948332959217570025	2	47124307505929254894195
	4	14	−2271655218574552381662735	3	72955475745396433550325
	5	8	−15734637384698230729299	4	72967622303350748309985
	6	13	−140876706167746254163647	5	47414037492407776412475
	7	11	−6029809415307042368265	6	19641996598864475451195
	8	15	−9466948716243617126925	6	2479525940882069502489
	9	12	−59056313896848453885	5	87922908810647850393
	10	12	−1194732873880501221	3	758359088382325275
	11	10	−1465252331444181	6	77129766093768975
	12	11	−440780211405	6	183586390214625
18	0	16	2824650747089425586152484375	0	−2329896471102350138203125
	1	16	15253114034282898165223415625	1	−26560819770566791575515625
	2	14	8962218708408279012965821875	1	−66408380156431092180684375
	3	14	12037097807676834994382158125	1	−95793913497035798008528425
	4	15	20402156697392380199935884525	2	−176464698144974099104278195
	5	15	11370339766947244262789322675	3	−217014300676187455171728375
	6	14	2107511053740623236455954261	2	−45254868642694448448927615
	7	14	514414248550848521925343659	4	−51021602204763628847836125
	8	16	320568097510659981999168855	7	−76120345271597706404420085
	9	16	30099680613168442180338825	8	−18026640750211258964286681
	10	13	193108558150711180896525	8	−1264692207546601870565709
	11	13	4562032403798021559267	7	−23424949863775646291025
	12	12	17607525453165335139	6	−186610006358507704275
	13	12	17629721383557285	8	−3334058907673648725
	14	0	0	8	−1102027252816125
20	0	18	−34846299923271027303905764459375	0	7982322432441532563075684375
	1	17	−104538899769813081911717293378125	1	100577262648763310294753623125
	2	18	−554101080726888055998921479405625	0	140817953273447264283235525500
	3	17	−427203420458215625218356777307125	1	462304855841998138828056037425
	4	17	−424991795182127332160223243959775	3	1976296094581464241686214934985
	5	16	−142960891137020917161335532343233	4	2888903038760151219665124511815
	6	17	−132635656222747471778421973692105	0	92279282449320631322903764260
	7	16	−21240778594401517343353144648929	2	132608379346222489393206193245
	8	18	−18530648765562341168546344180299	7	1062806956285521881259642166875
	9	17	−1335579126308922606873032987145	8	363219722456634799310301146505
	10	18	−241967021345882731513296783189	5	5098635722556771487174831527
	11	17	−6349188535732704774197483865	8	2836521325580179936283761059
	12	15	−42166127517040384427123523	7	55575365550609909735748935
	13	14	−221731883523034170402681	8	2095733015764960178015775
	14	14	−535691570519652576885	6	3533273075796511619025
	15	13	−17633793609271701	8	16760795964881617125

TABLE S4. Exact values of $k_n^{(m)} = \beta_{nm}/2^{h_{nm}}$. The $k_n^{(m)}$ are given by the recursion relation in Eq. (A3).

n	b_n	$b_n^{(0)}/b_n$
1	$3.5367765131532296837529725193892080452102 1 \times 10^{-02}$	1.21585
2	$-3.5275797460324077952779866056284818094054 3 \times 10^{-03}$	1.23513
3	$-1.7657313662486726337548767312635234514833 7 \times 10^{-03}$	1.50009
4	$4.1734667917685853760184598090005497412488 0 \times 10^{-03}$	0.96457
5	$1.0164712339943922520521204747703131420939 7 \times 10^{-02}$	1.12356
6	$-5.1361976913663510967072887054087448117509 5 \times 10^{-02}$	1.01382
7	$-3.1518396487744871221291175560688241359866 7 \times 10^{-01}$	1.10480
8	$3.2211643126550119991911330626465142307531 2 \times 10^{+00}$	0.99673
9	$3.6531937628296466190453947771430961001988 1 \times 10^{+01}$	1.06856
10	$-6.0465098828277951920784761252514896294639 7 \times 10^{+02}$	1.00082
11	$-1.1031953657663325537382591657907680201036 1 \times 10^{+04}$	1.05600
12	$2.7274138821174993576979698484546003513324 2 \times 10^{+05}$	0.99972
13	$7.2919760341999945854004547779982302411011 2 \times 10^{+06}$	1.04567
14	$-2.5108543893219687646026234106501615356227 2 \times 10^{+08}$	1.00000
15	$-9.2554804360131293951139955660916737487888 4 \times 10^{+09}$	1.03900
16	$4.2386788923871516310410106858238487109708 6 \times 10^{+11}$	0.99994
17	$2.0601681741478165786161123919957053405559 1 \times 10^{+13}$	1.03391
18	$-1.2107786412198530608482264161263015298977 0 \times 10^{+15}$	0.99996
19	$-7.5030885369902368427994798303115107159699 4 \times 10^{+16}$	1.03003
20	$5.5050395328302837052346619274132041952159 1 \times 10^{+18}$	0.99996
21	$4.2363932520730303865223285964246580644586 5 \times 10^{+20}$	1.02694
22	$-3.7953866537965248917470212253873818119049 9 \times 10^{+22}$	0.99997
23	$-3.5510348157215990870993190653079437587529 7 \times 10^{+24}$	1.02442
24	$3.8149279948019332814713459017526972065416 7 \times 10^{+26}$	0.99997
25	$4.2647089981370138590742158784409882649598 4 \times 10^{+28}$	1.02234
26	$-5.4116603344612400810732189231678610848274 4 \times 10^{+30}$	0.99998
27	$-7.1238745193485572468984355199106939773452 0 \times 10^{+32}$	1.02058
28	$1.0541756432314261511300663427115328800129 7 \times 10^{+35}$	0.99998

TABLE S5. Numerical values of the b_n, which give the explicit WKB series in Eq. (5), to 41 digits. The last column compares the exact b_n values with their asymptotic approximation in Eq. (6).

n	Q_n	m	k_{nm}
1	$3/2$	0	1
2	3^2	0	5
		1	2
3	3^4	0	88
		1	33
		2	6
4	$3^5 2$	0	1309
		1	476
		2	102
		3	12
5	3^6	0	5474
		1	1955
		2	460
		3	69
		4	6
6	$3^8 2$	0	294814
		1	104052
		2	26013
		3	4524
		4	522
		5	36
7	3^9	0	1387360
		1	485576
		2	126672
		3	24360
		4	3360
		5	315
		6	18
8	$3^{10} 2$	0	27018836
		1	9397856
		2	2530192
		3	523488
		4	81795
		5	9348
		6	738
		7	36
9	$3^{13} 5$	0	6073364440
		1	2102318460
		2	579949920
		3	126864045
		4	21748122
		5	2861595
		6	279180
		7	19035
		8	810
10	$3^{14} 28$	0	346648954960
		1	119534122400
		2	33618971925
		3	7684336440
		4	1415535660
		5	207151560
		6	23539950
		7	2003400
		8	120204
		9	4536

TABLE S6. Exact values of $L_{nm} = \frac{2 \times 3^m k_{nm}}{Q_n (2\alpha)^{3n/2}}$. The L_{nm} are used to construct the b_n in terms of the a_n via Eq. (B4).

n	G_n	g_{n0}	g_{n1}	g_{n2}	g_{n3}	g_{n4}	g_{n5}
0	9	1					
1	1	1					
2	1/72	5	11/192				
3	11/972	−1	93/640				
4	17/559872	−77	539/20	102829/86016			
5	23/5038848	119	−3289/48	28171999/430080			
6	29/1088391168	5083	−661089/160	6734014687/716800	49829732957/90832896		
7	1/1224440064	−43355	2931929/64	−10264192781/61440	492349052125069/1349517312		
8	41/176319369216	−164749/4	806113/15	−262775969173/983040	787570022698313/527155200	45866361756966241/355140108288	
9	47/2115832430592	3230513/27	−7446461/40	4267944409223/3686400	−1335041940357576377/120766464000	4907566420869344641093/98107454914560	
10	53/1371059415023616	58397735/3	−9015402055/256	89325797863511/344064	−955865010579864937/268369920	1407560427696573497146789/32702484971520	56201923399216345104411 41/1187588522115072

TABLE S7. The constants for writing the $b_n = \frac{(-1)^{\lfloor n/2 \rfloor}}{9\pi^n} G_n \sum_{m=0}^{\lfloor n/2 \rfloor} g_{nm} \left(\frac{\gamma^2}{\pi}\right)^{4m}$. This Table is reproduced from Ref. [41].

n	d_n
1	$5.3051647697298445256294587790838120678153 2 \times 10^{-02}$
2	$-1.4072386616991357144983258779128838736716 5 \times 10^{-03}$
3	$-2.9261081615006970204875168665100616048013 5 \times 10^{-03}$
4	$4.7263570020280663162544437416356988612065 0 \times 10^{-04}$
5	$1.6288573924261885762845891625035856031306 0 \times 10^{-02}$
6	$-4.3520826617623507760980420496523443132561 8 \times 10^{-03}$
7	$-4.9351130669750997776697143450568973305296 9 \times 10^{-01}$
8	$1.8378636150210120003924855459604494934269 2 \times 10^{-01}$
9	$5.6611199504354776878259877931775582706656 8 \times 10^{+01}$
10	$-2.7056833881000530484392544366707138241786 4 \times 10^{+01}$
11	$-1.6984174564625705379275429957068110662784 0 \times 10^{+04}$
12	$9.9161748884504910311421759670829565749170 9 \times 10^{+03}$
13	$1.1178146502188305582963920748038992450400 9 \times 10^{+07}$
14	$-7.7113761125600063318596054011560413041649 4 \times 10^{+06}$
15	$-1.4144286697914865052227898795489983082115 8 \times 10^{+10}$
16	$1.1257692963009760422779646726921128587421 0 \times 10^{+10}$
17	$3.1410707795564000120848866587498153081407 3 \times 10^{+13}$
18	$-2.8332224503612822178165935982021677003818 2 \times 10^{+13}$
19	$-1.1419068537210364505494898338468062871544 1 \times 10^{+17}$
20	$1.1511296256476172179588609358260683525317 4 \times 10^{+17}$
21	$6.4381166148128683696688348698327622202066 7 \times 10^{+20}$
22	$-7.1731322472436845046382885751071824195017 8 \times 10^{+20}$
23	$-5.3901801857694278979841370036636812463427 5 \times 10^{+24}$
24	$6.5774176280896618068445285963888154495053 5 \times 10^{+24}$
25	$6.4670915210247551049737567962580284301401 4 \times 10^{+28}$
26	$-8.5776572361838857811022579364897166251080 0 \times 10^{+28}$
27	$-1.0793763910220275233359435640074504971056 3 \times 10^{+33}$
28	$1.5461545272135932824678136214252342661606 7 \times 10^{+33}$

TABLE S8. Numerical values of the d_n coefficients of Eq. (10) to 41 digits.

n	p_n	m	r	k_{nmr}
1	1	1	1	−3
2	4	1	1	3
		2	1	24
			2	−3
3	5	1	1	9
		2	1	−24
			2	21
		3	1	−96
			2	12
			3	−5
4	10	1	1	15
		2	1	−72
			2	117
		3	1	96
			2	−84
			3	77
		4	1	1536
			2	−192
			3	80
			4	−45
5	11	1	1	21
		2	1	−120
			2	285
		3	1	288
			2	−468
			3	663
		4	1	−1536
			2	1344
			3	−1232
			4	1155
		5	1	−6144
			2	768
			3	−320
			4	180
			5	−117
6	14	1	1	27
		2	1	−168
			2	525
		3	1	480
			2	−1140
			3	2185
		4	1	−4608
			2	7488
			3	−10608
			4	13923
		5	1	6144
			2	−5376
			3	4928
			4	−4620
			5	4389
		6	1	49152
			2	−6144
			3	2560
			4	−1440
			5	936
			6	−663
7	15	1	1	33
		2	1	−216
			2	837
		3	1	672
			2	−2100
			3	5075
		4	1	−7680
			2	18240
			3	−34960
			4	58995
		5	1	18432
			2	−29952
			3	42432
			4	−55692
			5	69615
		6	1	−49152
			2	43008
			3	−39424
			4	36960
			5	−35112
			6	33649
		7	1	−196608
			2	24576
			3	−10240
			4	5760
			5	−3744
			6	2652
			7	−1989

n	p_n	m	r	k_{nmr}
8	22	1	1	39
		2	1	−264
			2	1221
		3	1	864
			2	−3348
			3	9765
		4	1	−10752
			2	33600
			3	−81200
			4	167475
		5	1	30720
			2	−72960
			3	139840
			4	−235980
			5	365769
		6	1	−147456
			2	239616
			3	−339456
			4	445536
			5	−556920
			6	672945
		7	1	196608
			2	−172032
			3	157696
			4	−147840
			5	140448
			6	−134596
			7	129789
		8	1	6291456
			2	−786432
			3	327680
			4	−184320
			5	119808
			6	−84864
			7	63648
			8	−49725
9	23	1	1	45
		2	1	−312
			2	1677
		3	1	1056
			2	−4884
			3	16687
		4	1	−13824
			2	53568
			3	−156240
			4	380835
		5	1	43008
			2	−134400
			3	324800
			4	−669900
			5	1239315
		6	1	−245760
			2	583680
			3	−1118720
			4	1887840
			5	−2926152
			6	4267305
		7	1	589824
			2	−958464
			3	1357824
			4	−1782144
			5	2227680
			6	−2691780
			7	3172455
		8	1	−6291456
			2	5505024
			3	−5046272
			4	4730880
			5	−4494336
			6	4307072
			7	−4153248
			8	4023459
		9	1	−25165824
			2	3145728
			3	−1310720
			4	737280
			5	−479232
			6	339456
			7	−254592
			8	198900
			9	−160225

m	r	k_{10mr}
1	1	51
2	1	−360
	2	2205
3	1	1248
	2	−6708
	3	26273
4	1	−16896
	2	78144
	3	−266992
	4	750915
5	1	55296
	2	−214272
	3	624960
	4	−1523340
	5	3275181
6	1	−344064
	2	1075200
	3	−2598400
	4	5359200
	5	−9914520
	6	16937305
7	1	983040
	2	−2334720
	3	4474880
	4	−7551360
	5	11704608
	6	−17069220
	7	23774985
8	1	−18874368
	2	30670848
	3	−43450368
	4	57028608
	5	−71285760
	6	86136960
	7	−101518560
	8	117380835
9	1	25165824
	2	−22020096
	3	20185088
	4	−18923520
	5	17977344
	6	−17228288
	7	16612992
	8	−16093836
	9	15646785
10	1	201326592
	2	−25165824
	3	10485760
	4	−5898240
	5	3833856
	6	−2715648
	7	2036736
	8	−1591200
	9	1281800
	10	−1057485

TABLE S9. This table yields the exact $F_{nmr} = (-1)^n \sqrt{\alpha^{3(m-n)}/2^{3n+p_m}}\, k_{nmr}$, which give the d_n in terms of the a_n and b_n via Eq. (B6). For all n $k_{n00} = 1$. Also $p_0 = 0$ and $p_{10} = 26$.

n	p_n	m	t_{nm}
1	0	0	1
2	1	0	1
3	1	0	1
		1	1133/8640
4	3	0	5
		1	1133/720
5	3	0	7
		1	7931/2160
		2	7692847/2322432
6	4	0	21
		1	1133/72
		2	869938199/26127360
7	4	0	33
		1	12463/384
		2	19237484431/174182400
		3	88834768437119/382588157952
8	7	0	429
		1	1134133/2160
		2	1904335381/777600
		3	1335014301312971/102478970880
9	7	0	715
		1	1134133/1080
		2	22968840271/3732480
		3	128974728548986789/2305776844800
		4	678896102406979847771/275078210125824
10	8	0	2431
		1	250393/60
		2	3018893473/103680
		3	5125598428842788291/13450364928000
		4	110379963562766171902 9/248334495252480

TABLE S10. The t_{nm} and p_n used to construct the d_n explicitly via $d_n = 2^{-p_n} \frac{(-1)^{n+1}}{(6\pi)^n} \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \left[-\left(\frac{\gamma^2}{\pi}\right)^4 \right]^m t_{nm}$.

n	h_n	h_{n+28}
1	$-1.6666666666666666666666666666666666666667 \times 10^{-1}$	$4.3597792139867817830689064847401493504677 1 \times 10^{10}$
2	$-3.7949936768772595298759737080571513371621 8 \times 10^{-3}$	$1.9165662210968919888413818753761272700698 6 \times 10^{11}$
3	$6.5966794641309569028804211437892451559163 4 \times 10^{-3}$	$-6.8684137395479781994872085642827738164695 6 \times 10^{10}$
4	$6.1048717775057908841636866060768338120137 9 \times 10^{-3}$	$-9.7400130682649005946297891251557734782321 4 \times 10^{12}$
5	$8.0125757303958425260090875147590612334011 6 \times 10^{-3}$	$-9.7053642490205574550499935758959625913645 3 \times 10^{13}$
6	$4.7647939213006470396773113447611876027354 1 \times 10^{-3}$	$-4.8843002029229554180228163155297804262359 7 \times 10^{14}$
7	$-2.5265321635203451740564300327770467537629 6 \times 10^{-3}$	$1.7347259237362370986017948396143113746110 6 \times 10^{14}$
8	$-2.0478341051757080762989013962431186027317 8 \times 10^{-2}$	$3.1457754869853068342605406623305777898237 8 \times 10^{16}$
9	$-4.7609408146136569537249450311596111958800 1 \times 10^{-2}$	$3.5349103678924177282081298229608536056422 5 \times 10^{17}$
10	$-5.7814481729986705528453106021456734683483 7 \times 10^{-2}$	$2.0039966352966247916909995101017905399696 0 \times 10^{18}$
11	$2.4844428564113942904012091276234766540169 3 \times 10^{-2}$	$-7.0678072715100552486896515840067653596302 0 \times 10^{17}$
12	$4.0735528906903338824313390611304555533338 9 \times 10^{-1}$	$-1.5955486976810786696566488013395905659371 8 \times 10^{20}$
13	$1.4811243113281266806927606320450064749438 6 \times 10^{+0}$	$-1.9959749075850439101468722705653933638672 7 \times 10^{21}$
14	$2.7413143251427321051337356932333498694892 4 \times 10^{+0}$	$-1.2586118737627845219675643156019407017029 4 \times 10^{22}$
15	$-1.0815511614907364748118158643913475256324 4 \times 10^{+0}$	$4.4141175654400268328427992483372727722885 5 \times 10^{21}$
16	$-3.5284718430031659769956158088486835438363 6 \times 10^{+1}$	$1.2139554497039805883560465755364384373709 0 \times 10^{24}$
17	$-1.7191627191014706270765154054149751525516 6 \times 10^{+2}$	$1.6731260307676762417439523863866638346074 0 \times 10^{25}$
18	$-4.2749283782196864759292945239978011729766 5 \times 10^{+2}$	$1.1615425095900379024809229123445034262057 7 \times 10^{26}$
19	$1.6168929237213699589144796140441274063966 5 \times 10^{+2}$	$-4.0550124529422331947877525899679769944864 9 \times 10^{25}$
20	$8.4434420981188083772952916037663109718175 1 \times 10^{+3}$	$-1.3347040942847129068977386596224361110741 0 \times 10^{28}$
21	$5.1943042726317121145400785388383383138143 5 \times 10^{+4}$	$-2.0094367022191309423588316096399285829627 0 \times 10^{29}$
22	$1.6221940354440185818041262690344837540715 6 \times 10^{+5}$	$-1.5229427064108849504359641573385790658876 8 \times 10^{30}$
23	$-5.9828125868910099743186240752926955320444 4 \times 10^{+4}$	$5.2963486837263545840012930981187499722335 2 \times 10^{29}$
24	$-4.6286886440529339849731316767997875125002 1 \times 10^{+6}$	$2.0557522926998180489245253124127153635990 3 \times 10^{32}$
25	$-3.4343262164527573432686993834369104961639 5 \times 10^{+7}$	$3.3566790238376453022211221595105353792287 3 \times 10^{33}$
26	$-1.2911299257553158654278008251888159589448 8 \times 10^{+8}$	$2.7577001615045536245498351725998141613217 4 \times 10^{34}$
27	$4.6827614587536587420051683780017304124789 3 \times 10^{+7}$	$-9.5594036858718668221506310451571184567057 3 \times 10^{33}$
28	$5.0155684324818169581974825798837849944818 9 \times 10^{+9}$	$-4.3208975197436393675754033123804387470923 5 \times 10^{36}$

TABLE S11. The numerical values of the known h_n coefficients, which give the subdominant corrections to the eigenvalues in Eq. (14), to 41 digits.

n	p_n	m	$q_{nm}(\mathbf{a})$
0	0	0	$1/2$
1	0	0	a_1
2	1	0	a_1
		1	$3a_1^2$
3	1	0	$5a_1^2$
		1	$2a_1^3$
4	2	0	$2(5a_1^2 + 6a_2)$
		1	$24a_1^3$
		2	$3a_1^4$
5	2	0	$44a_1^3 + 33a_1a_2 + 9a_3$
		1	$22a_1^4$
		2	$6a_1^5/5$
6	4	0	$7(44a_1^3 + 99a_1a_2 + 27a_3)$
		1	$9a_1(77a_1^3 + 42a_1a_2 + 18a_3)$
		2	$126a_1^5$
		3	$18a_1^6/5$
7	4	0	$17a_1(77a_1^3 + 126a_1a_2 + 54a_3)$
		1	$6a_1^2(119a_1^3 + 51a_1a_2 + 27a_3)$
		2	$306a_1^6/5$
		3	$36a_1^7/35$
8	5	0	$5[1309a_1^4/2 + 2142a_1^2a_2 + 459(2a_1a_3 + a_2^2) + 162a_4]$
		1	$60a_1^2(119a_1^3 + 153a_1a_2 + 81a_3)$
		2	$18a_1^3(85a_1^3 + 30a_1a_2 + 18a_3)$
		3	$72a_1^7$
		4	$27a_1^8/35$
9	5	0	$13685a_1^5 + 35190a_1^3a_2 + 6210(a_1a_2^2 + 3a_1^2a_3) + 1863(a_2a_3 + a_1a_4) + 243a_5$
		1	$92a_1^3(85a_1^3 + 90a_1a_2 + 54a_3)$
		2	$18a_1^4(46a_1^3 + 69a_1a_2/5 + 9a_3)$
		3	$828a_1^8/35$
		4	$6a_1^9/35$
10	6	0	$13[2737a_1^5 + 11730a_1^3a_2 + 6210(a_1a_2^2 + a_1^2a_3) + 1863(a_2a_3 + a_1a_4) + 243a_5]$
		1	$3[25415a_1^6 + 53820a_1^4a_2 + 8073(a_1^2a_2^2 + 4a_1^3a_3) + 2106(2a_1a_2a_3 + a_1^2a_4) + 243(a_3^2 + 2a_1a_5)]$
		2	$78a_1^4(230a_1^3 + 207a_1a_2 + 135a_3)$
		3	$(18a_1^5/5)(299a_1^3 + 78a_1a_2 + 54a_3)$
		4	$702a_1^9/35$
		5	$18a_1^{10}/175$

TABLE S12. The $P_{nm}(\mathbf{a}) = \frac{2}{\pi^n} 3^{-p_n} \left(\frac{\gamma^4}{18\pi}\right)^{\lceil n/2 \rceil + m} q_{nm}(\mathbf{a})$, for constructing the h_n in terms of the a_n via Eq. (B8).

n	Q_n	m	g_{nm}
0	1	0	6π
1	1	0	$-\pi$
2	12	0	$\pi - 4$
3	216	0	$30 - \pi$
4	$10368\pi^5$	0	$2\pi^4[480 + (\pi - 96)\pi]$
		1	11
5	$622080\pi^5$	0	$-4\pi^4[5280 + (\pi - 220)\pi]$
		1	1023
6	$5598720\pi^6$	0	$\pi^4\{\pi[27720 + (\pi - 420)\pi] - 147840\}$
		1	$231(93 - 7\pi)$
7	$940584960\pi^6$	0	$4\pi^4\{2199120 - \pi[99960 + (\pi - 714)\pi]\}$
		1	$77(299\pi - 19992)$
8	$361184624640\pi^{11}$	0	$32\pi^8\{87964800 + \pi(\pi[285600 + (\pi - 1120)\pi] - 15993600)\}$
		1	$-7392\pi^4[133280 + \pi(67\pi - 11960)]$
		2	-43702325
9	$19503969730560\pi^{11}$	0	$-32\pi^8\{1655337600 + \pi(\pi[695520 + (\pi - 1656)\pi] - 78825600)\}$
		1	$11088\pi^4[1650480 + \pi(101\pi - 36984)]$
		2	-5831603793
10	$1170238183833600\pi^{12}$	0	$32\pi^8\{\pi[15370992000 + \pi(\pi[1506960 + (\pi - 2340)\pi] - 301392000)] - 86077555200\}$
		1	$14784\pi^4\{107281200 - \pi[10817820 + \pi(152\pi - 98475)]\}$
		2	$9(6734014687\pi - 168468554020)$

TABLE S13. The exact $h_n = \frac{1}{6\pi Q_n}\sum_{m=0}^{\lfloor n/4 \rfloor} g_{nm}\gamma^{8m}$ values.

n	c_n
1	$1.1794472629109644823307844672761493353508\,5 \times 10^{-01}$
2	$6.3907747479637536645185084295141691852414\,4 \times 10^{-03}$
3	$1.0367570280641476005921988209193094220879\,0 \times 10^{-05}$
4	$-1.9722651160054945850366608218779930753838\,0 \times 10^{-03}$
5	$-4.8669405048775756837304225994008051704708\,6 \times 10^{-05}$
6	$1.5096528036904715973393035420309283381882\,0 \times 10^{-02}$
7	$-6.9705407534555619525224632909061671931433\,7 \times 10^{-05}$
8	$-6.6837105322845801942951147296815111533494\,0 \times 10^{-01}$
9	$-1.8649425523450633616632057453166622933406\,2 \times 10^{-04}$
10	$9.7953353541172825052370514444114211277668\,7 \times 10^{+01}$
11	$-1.3983931129086152342806848695407057694000\,1 \times 10^{+00}$
12	$-3.6136218035182901491524062188734919251126\,0 \times 10^{+04}$
13	$7.3866895847316023941028812522491733212744\,3 \times 10^{+02}$
14	$2.8163980596302064134672848570531826619942\,1 \times 10^{+07}$
15	$-8.4427919571139647527612399396117873975338\,7 \times 10^{+05}$
16	$-4.1213073641312831854385536133559790441664\,1 \times 10^{+10}$
17	$1.5809822455342911765513217637268482949501\,7 \times 10^{+09}$
18	$1.0389563843658474677353312724593390973032\,0 \times 10^{+14}$
19	$-4.9013311503708729161165083646739021552114\,9 \times 10^{+12}$
20	$-4.2271719136886180317633861639981191915465\,5 \times 10^{+17}$
21	$2.3619005455946528613910478292604820444068\,9 \times 10^{+16}$
22	$2.6371281243071150183601684786320865391403\,8 \times 10^{+21}$
23	$-1.7042912820894301678458706579013727489136\,8 \times 10^{+20}$
24	$-2.4204438338346585788445830153250773309001\,4 \times 10^{+25}$
25	$1.7760013509641864127315446306019660762195\,2 \times 10^{+24}$
26	$3.1590923603009903670750000956047373798901\,3 \times 10^{+29}$
27	$-2.5946790323977054014186796417199116686981\,4 \times 10^{+28}$
28	$-5.6983744249199675247622264792835280735402\,1 \times 10^{+33}$

TABLE S14. The numerical values of the c_n coefficients, which give the SWKB series in Eq. (22), to 41 digits.

n	m	Q_{nm}	t_{nmr}
0	0	1	1
1	0	$-\frac{7}{9}$	$\frac{1}{6}, -1$
2	0	$-\frac{7}{324}$	$\frac{7}{180}, -\frac{1}{3}, -\frac{1}{2}$
	1	$\frac{77}{622080}$	1
3	0	$\frac{1}{8748}$	$\frac{31}{36}, -\frac{49}{6}, -\frac{35}{2}, -7$
	1	$-\frac{7}{373248}$	$\frac{11}{9}, -\frac{31}{5}$
4	0	$-\frac{77}{314928}$	$\frac{127}{720}, -\frac{31}{18}, -\frac{49}{12}, -\frac{7}{3}, -\frac{7}{16}$
	1	$\frac{539}{100776960}$	$\frac{77}{36}, -\frac{31}{2}, -7$
	2	$-\frac{102829}{61917364224}$	1
5	0	$\frac{119}{11337408}$	$\frac{2555}{648}, -\frac{1397}{36}, -\frac{1705}{18}, -\frac{539}{9}, -\frac{385}{24}, -\frac{7}{4}$
	1	$-\frac{77}{1088391168}$	$\frac{5797}{36}, -\frac{25823}{20}, -833, -\frac{299}{2}$
	2	$\frac{1}{557256278016}$	$\frac{8740465}{6}, -\frac{28171999}{5}$
6	0	$-\frac{391}{1224440064}$	$\frac{1414477}{6480}, -\frac{232505}{108}, -\frac{127127}{24}, -\frac{31031}{9}, -\frac{49049}{48}, -\frac{637}{4}, -\frac{91}{8}$
	1	$\frac{23023}{26121388032}$	$\frac{23749}{1080}, -\frac{16337}{90}, -\frac{5831}{45}, -\frac{299}{9}, -\frac{67}{20}$
	2	$-\frac{1}{20061226008576}$	$\frac{3658758649}{36}, -\frac{8423427701}{15}, -\frac{6734014687}{50}$
	3	$\frac{49829732957}{127107927990337536}$	1
7	0	$\frac{667}{11019960576}$	$\frac{974729}{324}, -\frac{24046109}{810}, -\frac{3952585}{54}, -\frac{2161159}{45}, -\frac{527527}{36}, -\frac{75803}{30}, -\frac{1547}{6}, -13$
	1	$-\frac{29029}{176319369216}$	$\frac{199801}{648}, -\frac{1539367}{600}, -\frac{84847}{45}, -\frac{48139}{90}, -\frac{1541}{20}, -\frac{101}{20}$
	2	$\frac{1}{135413275557888}$	$\frac{469889146493}{252}, -\frac{1709955823303}{150}, -\frac{195286425923}{50}, -\frac{10264192781}{25}$
	3	$-\frac{1}{285992837978259456}$	$\frac{1445062255753}{3}, -\frac{492349052125069}{260}$
8	0	$-\frac{88711}{396718580736}$	$\frac{118518239}{38880}, -\frac{4873645}{162}, -\frac{24046109}{324}, -\frac{3952585}{81}, -\frac{2161159}{144}, -\frac{47957}{18}, -\frac{10829}{36}, -\frac{65}{3}, -\frac{13}{16}$
	1	$\frac{3857}{4760622968832}$	$\frac{6083665577}{25920}, -\frac{563638621}{288}, -\frac{347946599}{240}, -\frac{30485741}{72}, -\frac{10797787}{160}, -\frac{101101}{16}, -\frac{1463}{5}$
	2	$-\frac{19}{4874877920083968}$	$\frac{1925029729181}{144}, -\frac{7572661503199}{90}, -\frac{9569034870227}{300}, -\frac{71849349467}{15}, -\frac{96812199169}{320}$
	3	$\frac{133}{2573935541804335104}$	$\frac{10115435790271}{72}, -\frac{492349052125069}{624}, -\frac{41451053826227}{325}$
	4	$-\frac{45866361756966241}{805089598413969775656 96}$	1
9	0	$\frac{13079}{42845606719488}$	$\frac{132242905811}{11664}, -\frac{362547293101}{3240}, -\frac{14908480055}{54}, -\frac{73557047431}{405}, -\frac{12090957515}{216}, -\frac{600998671}{60}, -\frac{20957209}{18}, -\frac{278369}{3}, -\frac{39767}{8}, -\frac{1729}{12}$
	1	$-\frac{59983}{9521245937664}$	$\frac{7151586673}{46656}, -\frac{497201395793}{388800}, -\frac{3690923873}{3888}, -\frac{3621902713}{12960}, -\frac{198106337}{4320}, -\frac{20523503}{4320}, -\frac{42427}{135}, -\frac{869}{80}$
	2	$\frac{779}{175495605123022848}$	$\frac{38727960299665}{648}, -\frac{341258326450613}{900}, -\frac{66592671239743}{450}, -\frac{5532399908959}{225}, -\frac{1064934190859}{480}, -\frac{38351233459}{400}$
	3	$-\frac{1}{12635683568857645056}$	$\frac{34896808414179197}{72}, -\frac{18793455668666008799}{6240}, -\frac{226032596514415831}{325}, -\frac{1335041940357576377}{26000}$
	4	$\frac{1}{24152687952419093269 7088}$	$\frac{20685729152391774691}{18}, -\frac{49075664208693446 41093}{1105}$
10	0	$-\frac{55883}{4627325525704704}$	$\frac{14738950654477}{7776}, -\frac{36366799098025}{1944}, -\frac{19940101120555}{432}, -\frac{819966403025}{27}, -\frac{4045637608705}{432}, -\frac{60454787575}{36}, -\frac{4722132415}{24}, -\frac{48430525}{3}, -\frac{15310295}{16}, -\frac{475475}{12}, -\frac{7315}{8}$
	1	$\frac{2819201}{2056589122535424}$	$\frac{869568319543}{186624}, -\frac{100772357665}{2592}, -\frac{112271282921}{3888}, -\frac{33120835475}{3888}, -\frac{811596929}{576}, -\frac{64921285}{432}, -\frac{296989}{27}, -\frac{4345}{8}, -\frac{22385}{1536}$
	2	$-\frac{1927}{18953525353286467584}$	$\frac{156678665198704765}{9072}, -\frac{118585436346063 05}{108}, -\frac{5183487603274189}{120}, -\frac{66501705028099}{9}, -\frac{141636247384247}{192}, -\frac{728673435721}{16}, -\frac{46354850993}{32}$
	3	$\frac{47}{9097692169577504440 32}$	$\frac{714821720742057745}{144}, -\frac{59448686298841456405}{1872}, -\frac{1582228175600910817}{195}, -\frac{1335041940357576377}{1560}, -\frac{20337553416592871}{520}$
	4	$-\frac{1}{372641471265894581875 5072}$	$\frac{48611463508120670523 85}{36}, -\frac{11532781089042959906568 55}{1547}, -\frac{14075604276965734971467 89}{15470}$
	5	$\frac{562019233992163451044 1141}{20934699742540944894100494 09024}$	1

TABLE S15. The Q_{nm} and t_{nmr}, from $r=0$ on the left to $r=n-2m$ on the right, to construct the exact form of the c_n via $c_n = \sum_{m=0}^{\lfloor n/2 \rfloor} Q_{nm} \left(\frac{\gamma^4}{\pi^3}\right)^{2m} \sum_{r=0}^{n-2m} \frac{t_{nmr}}{\pi^r}$.

n	q_n
1	$5.0683361517998533712196733561650382242397 4 \times 10^{-01}$
2	$1.3141208283551074687436956088911206199664 4 \times 10^{-03}$
3	$3.1907603000726898746093001983982869330869 8 \times 10^{-03}$
4	$-7.3277738437169261985694618534249235112827 0 \times 10^{-04}$
5	$-8.9540713243899497816674659713209316177368 0 \times 10^{-03}$
6	$1.9264794305662933869422738643412405701387 7 \times 10^{-03}$
7	$1.8259966058424801636655887000321080627834 9 \times 10^{-01}$
8	$-7.8978646920174108780364481930051024760822 6 \times 10^{-02}$
9	$-1.5756364970576017999184881634798045592257 6 \times 10^{+01}$
10	$8.0019108206941955505528555712149959570809 8 \times 10^{+00}$
11	$3.7871155746986804192591842169829972515222 8 \times 10^{+03}$
12	$-2.4699474459189849453328664282819369922310 5 \times 10^{+03}$
13	$-2.0790405126153797179119434851539166121800 4 \times 10^{+06}$
14	$1.5985344585002466813971353993193001526411 3 \times 10^{+06}$
15	$2.2563064128546217433555265383038568527856 9 \times 10^{+09}$
16	$-2.0216034873438077543502731515214764133162 6 \times 10^{+09}$
17	$-4.3863542520357681253630907815832159813147 0 \times 10^{+12}$
18	$4.4719173068917019709728312881447103829661 3 \times 10^{+12}$
19	$1.4179357460979313988970142351759890689188 0 \times 10^{+16}$
20	$-1.6224663478644184805718748663667440643123 7 \times 10^{+16}$
21	$-7.1969375221009879862577552929181611987311 3 \times 10^{+19}$
22	$9.1301020452691035256061392281018462847308 3 \times 10^{+19}$
23	$5.4789453683151408007956215538746602953377 8 \times 10^{+23}$
24	$-7.6327108675217971955161889976049933560044 6 \times 10^{+23}$
25	$-6.0269098056476966956662012361574800254501 7 \times 10^{+27}$
26	$9.1461670742728461979131203979703897778011 0 \times 10^{+27}$
27	$9.2867591005499453603931715400266040240844 3 \times 10^{+31}$
28	$-1.5249090616182306032500166005993878631587 0 \times 10^{+32}$

TABLE S16. The numerical values of the q_n, which give the G SWKB series in Eq. (C2), to 41 digits.

n	m	C_{nm}	g_{nmr}
0	0	1	1
1	0	$\frac{7}{9}$	$\frac{1}{3}, 1$
2	0	$\frac{7}{162}$	$\frac{1}{45}, -\frac{1}{3}, \frac{1}{4}$
	1	$\frac{77}{622080}$	1
3	0	$-\frac{1}{2187}$	$\frac{2}{9}, -\frac{7}{3}, -\frac{35}{4}, \frac{7}{4}$
	1	$\frac{7}{186624}$	$\frac{11}{9}, \frac{31}{10}$
4	0	$\frac{77}{19683}$	$\frac{1}{90}, -\frac{1}{9}, -\frac{7}{24}, -\frac{7}{24}, \frac{7}{256}$
	1	$-\frac{539}{50388480}$	$\frac{11}{9}, -\frac{31}{2}, \frac{7}{2}$
	2	$-\frac{102829}{61917364224}$	1
5	0	$-\frac{119}{177147}$	$\frac{5}{81}, -\frac{11}{18}, -\frac{55}{36}, -\frac{77}{72}, -\frac{385}{768}, \frac{7}{256}$
	1	$\frac{77}{136048896}$	$\frac{187}{9}, -\frac{3689}{20}, -\frac{833}{4}, \frac{299}{16}$
	2	$-\frac{1}{557256278016}$	$\frac{8740465}{3}, \frac{28171999}{5}$
6	0	$\frac{391}{9565938}$	$\frac{691}{405}, -\frac{455}{27}, -\frac{1001}{24}, -\frac{1001}{36}, -\frac{7007}{768}, -\frac{637}{256}, \frac{91}{1024}$
	1	$-\frac{23023}{1632586752}$	$\frac{187}{135}, -\frac{527}{45}, -\frac{833}{90}, -\frac{299}{72}, \frac{67}{320}$
	2	$\frac{1}{10030613004288}$	$\frac{522679807}{9}, -\frac{8423427701}{15}, \frac{6734014687}{100}$
	3	$\frac{49829732957}{127107927990337536}$	1
7	0	$-\frac{667}{43046721}$	$\frac{952}{81}, -\frac{46988}{405}, -\frac{7735}{27}, -\frac{17017}{90}, -\frac{17017}{288}, -\frac{10829}{960}, -\frac{1547}{768}, \frac{13}{256}$
	1	$\frac{29029}{2754990144}$	$\frac{391}{81}, -\frac{12121}{300}, -\frac{2737}{90}, -\frac{6877}{720}, -\frac{1541}{640}, \frac{101}{1280}$
	2	$-\frac{1}{16926659444736}$	$\frac{15157714403}{63}, -\frac{244279403329}{150}, -\frac{195286425923}{200}, \frac{10264192781}{200}$
	3	$\frac{1}{142996418989129728}$	$\frac{1445062255753}{3}, \frac{492349052125069}{520}$
8	0	$\frac{88711}{387420489}$	$\frac{3617}{1215}, -\frac{2380}{81}, -\frac{11747}{162}, -\frac{7735}{162}, -\frac{17017}{1152}, -\frac{1547}{576}, -\frac{1547}{4608}, -\frac{65}{1536}, \frac{13}{16384}$
	1	$-\frac{3857}{148769467776}$	$\frac{2971991}{405}, -\frac{1103011}{18}, -\frac{2739737}{60}, -\frac{983411}{72}, -\frac{1542541}{640}, -\frac{101101}{256}, \frac{1463}{160}$
	2	$\frac{19}{304679870005248}$	$\frac{15157714403}{18}, -\frac{244279403329}{45}, -\frac{1367004981461}{600}, -\frac{71849349467}{120}, \frac{96812199169}{5120}$
	3	$-\frac{133}{2573935541804335104}$	$\frac{1445062255753}{9}, -\frac{492349052125069}{312}, \frac{41451053826227}{325}$
	4	$-\frac{45866361756966241}{805089598413969775656 96}$	1
9	0	$-\frac{13079}{10460353203}$	$\frac{2017882}{729}, -\frac{11064403}{405}, -\frac{1820105}{27}, -\frac{35934073}{810}, -\frac{23661365}{1728}, -\frac{4732273}{1920}, -\frac{676039}{2304}, -\frac{39767}{1536}, -\frac{39767}{16384}, \frac{1729}{49152}$
	1	$\frac{59983}{74384733888}$	$\frac{873103}{729}, -\frac{242892719}{24300}, -\frac{7222943}{972}, -\frac{28518919}{12960}, -\frac{6390527}{17280}, -\frac{2931929}{69120}, -\frac{42427}{8640}, \frac{869}{10240},$
	2	$-\frac{779}{2742118830047232}$	$\frac{75788572015}{81}, -\frac{2687073436619}{450}, -\frac{2148150685153}{900}, -\frac{790342844137}{1800}, -\frac{1064934190859}{15360}, \frac{38351233459}{25600}$
	3	$\frac{1}{3158920892214411264}$	$\frac{1125703497231587}{9}, -\frac{2684779381238001257}{3120}, -\frac{226032596514415831}{650}, \frac{1335041940357576377}{104000}$
	4	$-\frac{1}{2415268795241909326970 88}$	$\frac{20685729152391774691}{9}, \frac{490756642086934464109 3}{1105}$
10	0	$\frac{55883}{282429536481}$	$\frac{28112371}{243}, -\frac{277458775}{243}, -\frac{608542165}{216}, -\frac{100105775}{54}, -\frac{1976374015}{3456}, -\frac{118306825}{1152}, -\frac{37182145}{3072}, -\frac{1562275}{1536}, -\frac{2187185}{32768}, -\frac{475475}{98304}, \frac{7315}{131072}$
	1	$-\frac{2819201}{8033551259904}$	$\frac{26537929}{1458}, -\frac{12302815}{81}, -\frac{54846743}{486}, -\frac{64815725}{1944}, -\frac{6390527}{1152}, -\frac{2094235}{3456}, -\frac{42427}{864}, -\frac{4345}{1024}, \frac{22385}{393216}$
	2	$\frac{1927}{148074416822550528}$	$\frac{76540627844995}{567}, -\frac{23206543316255}{27}, -\frac{40814863017907}{120}, -\frac{2145216291229}{36}, -\frac{20233749626321}{3072}, -\frac{728673435721}{1024}, \frac{46354850993}{4096}$
	3	$-\frac{47}{113721152119718805504}$	$\frac{5628517486157935}{9}, -\frac{1917699558027143755}{468}, -\frac{226032596514415831}{195}, -\frac{1335041940357576377}{6240}, \frac{20337553416592871}{4160}$
	4	$\frac{1}{130424514943063103656427 52}$	$\frac{486114635081206705238 5}{9}, -\frac{115327810890429599065685 5}{221}, \frac{140756042769657349714678 9}{4420}$
	5	$\frac{56201923399216345104411 41}{209346997425409448941004940902 4}$	1

TABLE S17. The C_{nm} and g_{nmr}, from $r = 0$ on the left to $r = n - 2m$ on the right, to construct the exact form of the q_n via $q_n = \sum_{m=0}^{\lfloor n/2 \rfloor} C_{nm} \left(\frac{\gamma^4}{\pi^3}\right)^{2m} \sum_{r=0}^{n-2m} \frac{g_{nmr}}{\pi^r}$.

3. NUMERICAL VALUES OF D SUMS

j	L	$D_L(z_j^2)$
0	2	1.18969077220200760959320513711674634073387
1	4	1.02306209111690103696010676599679697314490
2	6	1.00844219769426948586428708991107637336584
3	10	1.00431985323714094021023448659751750054629
4	12	1.00261605751525986295095876508991910070541
5	14	1.00175213109052792117656438364821563901564
6	16	1.00125483342767965871637398009070330157764
7	18	1.00094267928153460489963857081458647886944
8	20	1.00073400154519760196140033615232528333767
9	22	1.00058765314861704715613205941112351508087
10	24	1.00048107613023636243572927451368204649824

TABLE S18. The optimally truncated, via least addition, $D_{M'}(z_j^2)$ from Eq. (10) for the first 11 eigenvalues. We give the order, L, at which we truncate the sum for each eigenvalue.

4. SUBDOMINANT CORRECTIONS TO EIGENVALUES AND SUMS

j	L	$\epsilon_L^{\mathrm{SD}}(j+1/2)$
0	2	$4.98423055820686668764743454553873303393824 7 \times 10^{-02}$
1	7	$-4.25176939431772011755051955474252988393987 \times 10^{-03}$
2	11	$2.28025923532910565079832354657375019422319 \times 10^{-04}$
3	19	$-1.12467805686987628050920515996642836276596 \times 10^{-05}$
4	23	$5.34360714254869008273523755935775013434288 \times 10^{-07}$
5	27	$-2.48624007875120853182790354926761733555100 \times 10^{-08}$
6	31	$1.14140669578417918234184467155693160274292 \times 10^{-09}$
7	35	$-5.19166517060196160758140793578347160291439 \times 10^{-11}$
8	39	$2.34538047637032727503413102026053294780464 \times 10^{-12}$
9	43	$-1.05403481094866062110589677665327759720858 \times 10^{-13}$
10	47	$4.71745228714117478013995241397867506743425 \times 10^{-15}$

TABLE S19. The ϵ^{SD} values calculated via optimal truncation by least addition from Eq. (14).

N	Asym. SD
0	$-4.58617123576080125375621961710 48 \times 10^{-02}$
1	$4.03383052422695642258807429901 32 \times 10^{-03}$
2	$-2.17287572832808688462094047456 98 \times 10^{-04}$
3	$1.07361692574617771844776631541 36 \times 10^{-05}$
4	$-5.10589761517473014556092408225 57 \times 10^{-07}$
5	$2.37706614234889938170255235409 51 \times 10^{-08}$
6	$-1.09173444206946124295243568961 48 \times 10^{-09}$
7	$4.96721566364311061346154927041 94 \times 10^{-11}$
8	$-2.24449292884304533130329248144 94 \times 10^{-12}$
9	$1.00887496179852601754557712506 54 \times 10^{-13}$
10	$-4.51598360000181659253407230284 68 \times 10^{-15}$

TABLE S20. The asymptotic approximation to E^{SD} in Eq. (27).

5. ERRORS AROUND THE ORDER OF OPTIMAL TRUNCATION FOR VARIOUS APPROXIMATIONS

			Error		
j	M	M'	WKB	CWKB	$\epsilon_{L-1}^{\mathrm{WKB}} + \epsilon_{M'}^{\mathrm{SD}}$
5	11	25	$2.4857439 \times 10^{-08}$	$-4.9613663 \times 10^{-12}$	$-3.3805929 \times 10^{-14}$
	12	26	$2.4862367 \times 10^{-08}$	$-3.3805755 \times 10^{-14}$	$-3.3805743 \times 10^{-14}$
	13	27	$2.4866722 \times 10^{-08}$	$4.3213208 \times 10^{-12}$	$-3.3805755 \times 10^{-14}$
	14	28	$2.4861765 \times 10^{-08}$	$-6.3605332 \times 10^{-13}$	$-3.3805994 \times 10^{-14}$
	15	29	$2.4855724 \times 10^{-08}$	$-6.6769826 \times 10^{-12}$	$-3.3806373 \times 10^{-14}$
6	13	29	$-1.1413480 \times 10^{-09}$	$5.8688085 \times 10^{-14}$	$1.0688828 \times 10^{-15}$
	14	30	$-1.1414056 \times 10^{-09}$	$1.0688829 \times 10^{-15}$	$1.0688829 \times 10^{-15}$
	15	31	$-1.1414559 \times 10^{-09}$	$-4.9202168 \times 10^{-14}$	$1.0688829 \times 10^{-15}$
	16	32	$-1.1414014 \times 10^{-09}$	$5.2885839 \times 10^{-15}$	$1.0688828 \times 10^{-15}$
	17	33	$-1.1413387 \times 10^{-09}$	$6.7974250 \times 10^{-14}$	$1.0688826 \times 10^{-15}$
7	15	33	$5.1915956 \times 10^{-11}$	$-6.9542603 \times 10^{-16}$	$-1.8636461 \times 10^{-17}$
	16	34	$5.1916633 \times 10^{-11}$	$-1.8636461 \times 10^{-17}$	$-1.8636461 \times 10^{-17}$
	17	35	$5.1917218 \times 10^{-11}$	$5.6615805 \times 10^{-16}$	$-1.8636461 \times 10^{-17}$
	18	36	$5.1916607 \times 10^{-11}$	$-4.4844240 \times 10^{-17}$	$-1.8636461 \times 10^{-17}$
	19	37	$5.1915934 \times 10^{-11}$	$-7.1796910 \times 10^{-16}$	$-1.8636461 \times 10^{-17}$
8	17	37	$-2.3453722 \times 10^{-12}$	$8.2487009 \times 10^{-18}$	$2.7529485 \times 10^{-19}$
	18	38	$-2.3453802 \times 10^{-12}$	$2.7529485 \times 10^{-19}$	$2.7529485 \times 10^{-19}$
	19	39	$-2.3453870 \times 10^{-12}$	$-6.5635279 \times 10^{-18}$	$2.7529485 \times 10^{-19}$
	20	40	$-2.3453801 \times 10^{-12}$	$3.8133805 \times 10^{-19}$	$2.7529485 \times 10^{-19}$
	21	41	$-2.3453727 \times 10^{-12}$	$7.7784440 \times 10^{-18}$	$2.7529485 \times 10^{-19}$
9	20	41	$1.0540348 \times 10^{-13}$	$-3.7822120 \times 10^{-21}$	$7.6509652 \times 10^{-20}$
	21	42	$1.0540356 \times 10^{-13}$	$7.6509652 \times 10^{-20}$	$7.6509652 \times 10^{-20}$
	22	43	$1.0540348 \times 10^{-13}$	$-3.1950767 \times 10^{-21}$	$7.6509652 \times 10^{-20}$
	23	44	$1.0540340 \times 10^{-13}$	$-8.5824701 \times 10^{-20}$	$7.6509652 \times 10^{-20}$
	24	45	$1.0540349 \times 10^{-13}$	$1.2535630 \times 10^{-20}$	$7.6509652 \times 10^{-20}$
10	22	45	$-4.7174522 \times 10^{-15}$	$4.9915530 \times 10^{-23}$	$-8.9559882 \times 10^{-22}$
	23	46	$-4.7174532 \times 10^{-15}$	$-8.9559882 \times 10^{-22}$	$-8.9559882 \times 10^{-22}$
	24	47	$-4.7174523 \times 10^{-15}$	$2.5743333 \times 10^{-23}$	$-8.9559882 \times 10^{-22}$
	25	48	$-4.7174513 \times 10^{-15}$	$9.5995525 \times 10^{-22}$	$-8.9559882 \times 10^{-22}$
	26	49	$-4.7174524 \times 10^{-15}$	$-1.1529099 \times 10^{-22}$	$-8.9559882 \times 10^{-22}$

TABLE S21. The effect of varying the WKB order M in Eqs. (5) & (15), when approximating eigenvalues. We calculate the subdominant correction from Eq. (14). We vary M around the order of optimal truncation via least addition, i.e. from $L-2$ to $L+2$. In the last column we keep M fixed at $L-1$ and vary the order of the subdominant correction from $M' = L-2$ to $L+2$.

		Error		
N	M	SWKB	CSWKB	Hyper
5	10	-2.37643×10^{-08}	6.39009×10^{-12}	3.16565×10^{-13}
	11	-2.37643×10^{-08}	6.37835×10^{-12}	3.15656×10^{-13}
	12	-2.37764×10^{-08}	-5.75696×10^{-12}	1.81445×10^{-12}
	13	-2.37764×10^{-08}	-5.74704×10^{-12}	1.81034×10^{-12}
	14	-2.37613×10^{-08}	9.38587×10^{-12}	-8.33870×10^{-12}
6	12	1.09166×10^{-09}	-7.54974×10^{-14}	-5.19280×10^{-15}
	13	1.09166×10^{-09}	-7.53647×10^{-14}	-5.18497×10^{-15}
	14	1.09180×10^{-09}	6.51118×10^{-14}	-1.42534×10^{-14}
	15	1.09180×10^{-09}	6.49948×10^{-14}	-1.42274×10^{-14}
	16	1.09164×10^{-09}	-9.36185×10^{-14}	6.07308×10^{-14}
7	14	-4.96713×10^{-11}	8.93029×10^{-16}	7.33246×10^{-17}
	15	-4.96713×10^{-11}	8.91385×10^{-16}	7.31437×10^{-17}
	16	-4.96729×10^{-11}	-7.46380×10^{-16}	1.14050×10^{-16}
	17	-4.96729×10^{-11}	-7.45098×10^{-16}	1.13863×10^{-16}
	18	-4.96712×10^{-11}	9.74480×10^{-16}	-4.83323×10^{-16}
8	18	2.24450×10^{-12}	8.64582×10^{-18}	-8.91165×10^{-19}
	19	2.24450×10^{-12}	8.63167×10^{-18}	-8.89917×10^{-19}
	20	2.24448×10^{-12}	-1.04281×10^{-17}	4.09494×10^{-18}
	21	2.24448×10^{-12}	-1.04115×10^{-17}	4.08861×10^{-18}
	22	2.24451×10^{-12}	1.86180×10^{-17}	-1.39214×10^{-17}
9	20	-1.00888×10^{-13}	-1.01186×10^{-19}	6.15646×10^{-21}
	21	-1.00888×10^{-13}	-1.01030×10^{-19}	6.14869×10^{-21}
	22	-1.00887×10^{-13}	1.13499×10^{-19}	-3.66415×10^{-20}
	23	-1.00887×10^{-13}	1.13327×10^{-19}	-3.65911×10^{-20}
	24	-1.00888×10^{-13}	-1.86781×10^{-19}	1.11856×10^{-19}
10	22	4.51598×10^{-15}	1.20126×10^{-21}	-2.04314×10^{-23}
	23	4.51598×10^{-15}	1.19954×10^{-21}	-2.03952×10^{-23}
	24	4.51598×10^{-15}	-1.24213×10^{-21}	3.52168×10^{-22}
	25	4.51598×10^{-15}	-1.24033×10^{-21}	3.51755×10^{-22}
	26	4.51599×10^{-15}	1.94646×10^{-21}	-9.29020×10^{-22}

TABLE S22. How the SWKB [Eq. (22)], CSWKB [Eq. (28)], and hyperasymptotic [Eq. (29)] energies vary with order. For all the approximations we vary the order around the even order of least addition for SWKB.