

DFT: A Theory Full of Holes?

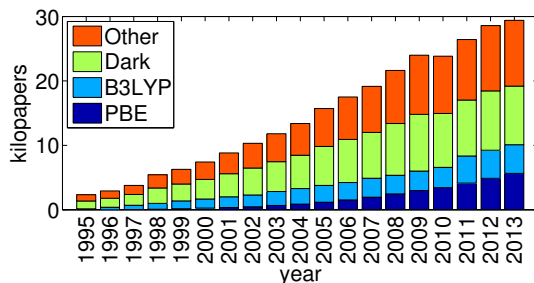
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This article is a rough, quirky overview of both the history and present state of the art of density functional theory. The field is so huge that no attempt to be comprehensive is made. We focus on the underlying exact theory, the origin of approximations, and the tension between empirical and non-empirical approaches. Many ideas are illustrated on the exchange energy and hole. Features unique to this article include how approximations can be systematically derived in a non-empirical fashion and a survey of warm dense matter.

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I. WHAT IS THIS ARTICLE ABOUT?

The popularity of density functional theory (DFT) as an electronic structure method is unparalleled, with applications that stretch from biology[1] to exoplanets[2]. However, its quirks of logic and diverse modes of practical application have led to disagreements on many fronts and from many parties. Developers of DFT are guided by many different principles, while applied practitioners (a.k.a. users) are suspicious of DFT for reasons both practical (*how can I pick a functional with so many choices?*[3]) and cultural (*with so many choices, why would I call this first-principles?*).

A modern DFT calculation[4] begins with the purchase of a computer, which might be as small as a laptop, and a quantum chemical code. Next, a basis set is chosen, which assigns predetermined functions to describe the electrons on each atom of the molecule being studied. Finally, a DFT approximation to something called the exchange-correlation energy (XC) is chosen, and the code starts running. For each guess of the nuclear positions, the code calculates an approximate energy[4]. A geometry optimization should find the minimum energy configuration. With variations on this theme[5, 6], one can read out all molecular geometries, dissociation energies, reaction barriers, vibrational frequencies, etc. A modern desktop may do a calculation for a 100-atom system within a day. A careful user will repeat the most im-

portant parts of the calculation with bigger basis sets, to check that answers don't change significantly.

II. WHERE DOES DFT COME FROM?

Although DFT's popularity has skyrocketed since applications to chemistry became useful and routine, its roots stretch back much further[7–9].

Ye olde DFT: Developed without reference to the Schrödinger equation[10], Thomas-Fermi (TF) theory[11–13] was the first DFT. It is pure DFT, relying only on the electronic density, $\rho(\mathbf{r})$, as input. The kinetic energy was approximated as that of a uniform electron gas, while the repulsion of the electrons was modeled with the classical electrostatic Coulomb repulsion, again depending only on the electronic density as an input.

Mixing in orbitals: John Slater was a master of electronic structure whose work foreshadowed the development of DFT. In particular, his X_α method[14] approximates the interactions of electrons in ground-state systems and improved upon Hartree-Fock (HF) [15, 16], one of the simplest ways to capture the Pauli exclusion principle. One of Slater's great insights was the importance of *holes*, a way of describing the depressed probability of finding electrons close to one another. Ahead of his time, Slater's X_α included focus on the hole, satisfied exact conditions like sum rules, and considered of the

degree of localization present in the system of interest.

A great logical leap: Although Slater’s methods provided an improvement upon HF, it was not until 1964 that Hohenberg and Kohn formulated their famous theorems[17], which now serve as the foundation of DFT: (i) the ground-state properties of an electronic system are completely determined by $\rho(\mathbf{r})$, and (ii) there is a one-to-one correspondence between the external potential and the density.

We write this by splitting the energy into two pieces:

$$E_{\text{elec}}[\textit{density}] = F[\textit{density}] + \textit{NucAtt}, \quad (1)$$

where E_{elec} is the total energy of the electrons, F is the sum of their exact quantum kinetic and electron-electron repulsion energies, and \textit{NucAtt} is their attraction to the nuclei in the molecule being calculated. Square brackets [] denote some (very complex) dependence on the one-electron density, $\rho(\mathbf{r})$, which gives the relative probability of finding an electron in a small chunk of space around the point \mathbf{r} . F is the same for all electronic systems, and so is called universal. For any given molecule, your computer simply finds $\rho(\mathbf{r})$ that minimizes E_{elec} above. Compare this to the variational principle in regular quantum mechanics. Instead of spending forever searching lots of wavefunctions that depend on all $3N$ electronic coordinates, you just search over one-electron densities, which have only 3 coordinates (and spin).

The pesky thing about the Hohenberg-Kohn theorems, however, is that they tell us such things exist without telling us how to find them. This means that to actually use DFT, we must approximate $F[\textit{density}]$. We recognize that the old TF theory did precisely this, with very crude approximations for the two main contributions to F :

$$F[\textit{density}] \sim \int d^3r \rho^{5/3}(\mathbf{r}) + \textit{CoulRep}, \quad (TF) \quad (2)$$

where we’ve not bothered with constants, etc. The first term is an approximation to the kinetic energy as a simple integral over the density. It is a local approximation, since the contribution at any point comes from only the density at that point. The other piece is the self-repulsion among electrons, which is simply modeled as the classical electrostatic repulsion, often called their Hartree energy or the direct Coulomb energy. Such simple approximations are typically good to within about 10% of the electronic energy, but bonds are a tiny fraction of this, and so are not accurate in such a crude theory[18].

A great calculational leap: Kohn and Sham proposed rewriting the universal functional in order to approximate only a small piece of the energy. They mapped the interacting electronic system to a fake non-interacting system with the same $\rho(\mathbf{r})$. This requires changing the external potential, so these aloof, non-interacting electrons produce the same density as their interacting cousins. The universal functional can now be broken into new pieces. Where in the interacting system, we had kinetic energy and electron-electron interaction terms, in

the Kohn-Sham (KS) system, we write the functional

$$F = \textit{OrbKE} + \textit{CoulRep} + \textit{XC} \quad (3)$$

where \textit{OrbKE} is the kinetic energy of the fake KS electrons. \textit{XC} contains all the rest, which includes both kinetic and potential pieces. Although it is small compared to the total, ‘nature’s glue’ [19] is critical to getting chemistry and physics right. The X part is (essentially) the Fock exchange from a HF calculation, while C is the correlation energy, i.e., that part that traditional methods such as coupled cluster usually get very accurately[20].

When you minimize this new expression for the energy, you find a set of orbital equations, the celebrated KS equations. They are almost identical to Hartree-Fock equations, and this showed that Slater’s idea could be made exact (if the exact functional were known). The genius of the KS scheme is that, because it calculates orbitals and gives their kinetic energy, only \textit{XC} , a small fraction of the total energy, needs to be approximated as a density functional. The KS scheme usually produces excellent self-consistent densities, even with simple approximations like LDA, but approximate potentials for this non-interacting KS system are typically very different from the exact KS potential (Fig. 1).

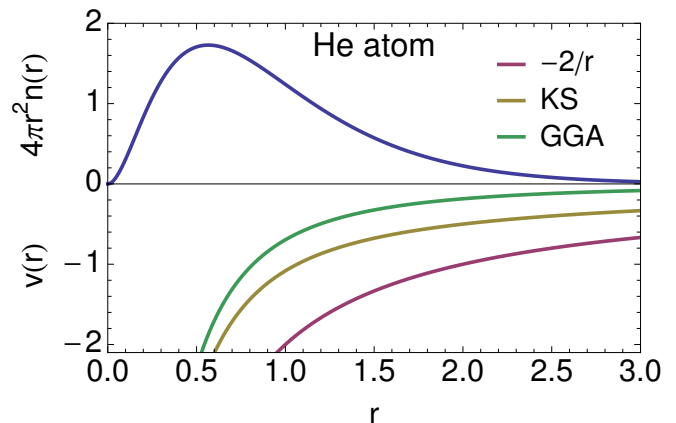


FIG. 1. Radial densities and potentials for the He atom (energies in Hartree, distances in Bohr). The pink line is $-2/r$, the attraction of real electrons to the nucleus. The yellow is the *exact* KS potential. Two fake electrons in the $1s$ orbital of this potential have the same ground-state density as real He. The green is the potential of a typical approximation which, although inaccurate, yields a highly accurate density.

Popular approximations for XC: Despite the overwhelming number of approximations available in the average DFT code, most calculations rely on a few of the most popular approximations. The sequence of these approximations is

$$\begin{aligned} \textit{XC} &\sim \textit{XC}^{\text{unif}}(\rho) && (\textit{LDA}) \\ &\sim \textit{XC}^{\text{GGA}}(\rho, |\nabla\rho|) && (\textit{GGA}) \\ &\sim a(\textit{X} - \textit{X}^{\text{GGA}}) + \textit{XC}^{\text{GGA}} && (\textit{hybrid}) \end{aligned} \quad (4)$$

The first was the third major step from the mid-60s and was invented in the KS paper[21]. It was the mainstay

of solid-state calculations for a generation, and remains popular for some specific applications even today. It is (almost) never used in quantum chemistry, as it typically overbinds by about 1eV/bond. The local density approximation (LDA)[21] assumes that the XC energy depends on the density at each position only, and that dependence is the same as in a uniform electron gas.

Adding another level of complexity leads to the more accurate generalized gradient approximations (GGAs)[22, 23], which use information about both the density and its gradient at each point. Hybrid approximations mix a fraction (a) of exact exchange with a GGA[24]. These maneuvers beyond the GGA usually increase the accuracy of certain properties with an affordable increase in computational cost[25]. (Meta-GGAs try to use a dependence on the KS kinetic energy density to avoid calculating the Fock exchange of hybrids[26, 27], which can be very expensive for solids.)

Fig. 2 shows that the two most popular functionals, PBE[28, 29] and B3LYP[24, 30], comprise a large fraction of DFT citations each year (about 2/3), even though they are now cited only about half the time they are used. PBE is a GGA, while B3LYP is a hybrid[24]. As a method tied to Hartree-Fock, quantum chemists' old stomping grounds, and one with typically higher accuracy than PBE, B3LYP is more often a chemist's choice. PBE's more systematic errors, mathematical rationale, and lack of costly exact exchange, have made it most popular in solid-state physics and materials science. In reality, both are used in both fields and many others as well.

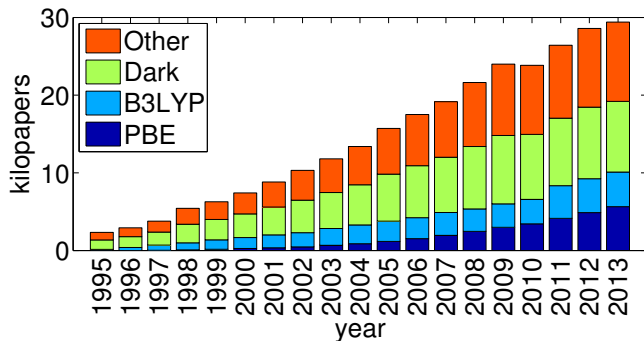


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Cultural wars: The LDA was defined by Kohn and Sham in 1965; there is no controversy about how it was designed. On the other hand, adding complexity to functional approximations demands choices about how to take the next step. Empirical functional developers fit their approximations to sets of highly accurate reference

data on atoms and molecules. Non-empirical developers use exact mathematical conditions on the functional and rely on reference systems like the uniform and slowly-varying electron gases. The PBE GGA is the most popular non-empirical approximation, while the most popular empirical functional approximation is the B3LYP hybrid. Modern DFT conferences usually include debates about the morality of this kind of empiricism.

Both philosophies have been incredibly successful, as shown by their large followings among developers and users, but each of these successes is accompanied by failures. No single approximation works well enough for every property of every material of interest. Many users sit squarely and pragmatically in the middle of the two factions, taking what is best from both of their accomplishments and insights. Often, empiricists and non-empiricists find themselves with similar end products, a good clue that something valuable has been created with the strengths of both.

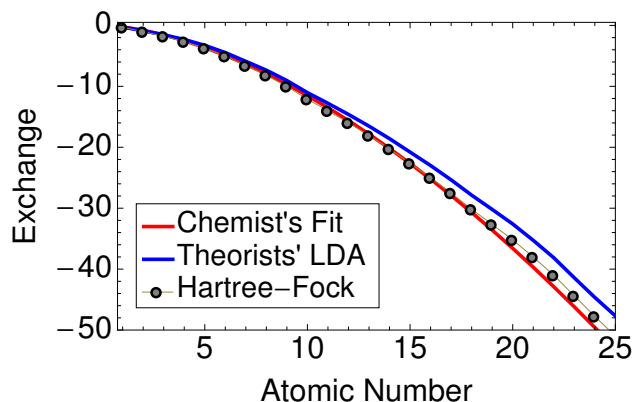


FIG. 3. Exchange energy (in Hartrees) of atoms from a HF calculation as a function of Z , atomic number, and two LDA X calculations, one with the theoretical asymptote, the other fitted.

To illustrate this idea, we give a brief allegory from an alternative universe. Since at least the 1960s, accurate HF energies of atoms have been available due to the efforts of Charlotte Froese Fischer and others[31, 32]. A bright young chemistry student plots these X energies as a function of Z , the atomic number, and notices they behave roughly as $Z^{5/3}$, as in Fig. 3. She's an organic chemistry student, and mostly only cares about main-group elements, so she fits the curve by choosing a constant to minimize the error on the first 18 elements, finding $E_x = -0.25Z^{5/3}$. Much later, she hears about KS DFT, and the need to approximate the XC energy. A little experimentation shows that if

$$X^{\text{opt}} = C_0 \int d^3r \rho^{4/3}(\mathbf{r}), \quad (5)$$

this goes as $Z^{5/3}$ when Z is large, and choosing $C_0 = -0.80$ makes it agree with her fit.

In our alternate timeline, a decade later, Paul Dirac, a very famous physicist, proves[33] that for a uniform gas, $C_0 = A_x = -(3/4)(3/\pi)^{1/3} = -0.738$. Worse still, Julian Schwinger proves[34] that inserting the TF density into Dirac’s expression becomes exact as $Z \rightarrow \infty$, so that $E_x \rightarrow -0.2201Z^{5/3}$. Thus theirs is the ‘correct’ LDA for X, and our brave young student should bow her head in shame.

Or should she? If we evaluate the mean absolute errors in exchange for the first 20 atoms, her functional is significantly better than the ‘correct’ one[35]. If lives depend on the accuracy for those 20 atoms, which would you choose[36]?

This simple fable contains the seeds of our actual cultural wars in DFT derivations:

(i) An intuitive, inspired functional need not wait for an official derivation. One parameter might be extracted by fitting, and later derived.

(ii) A fitted functional will usually be more accurate than the derived version for the cases where it was fitted. The magnitude of the errors will be smaller, but less systematic.

(iii) The fitted functional will miss universal properties of a derived functional. We see in Sec. VI that the correct LDA for exchange is a universal limit of *all* systems, not just atoms.

(iv) If you want to add the next correction to LDA, starting with the wrong constant will make life very difficult (see Sec. VI).

III. WHAT’S AT THE FOREFRONT?

Accurate Gaps: Calculating accurate energy gaps and self-interaction errors are notorious difficulties within DFT[37]. Self-interaction error (SIE) stems from spurious interaction of an electron with itself in the Coulomb repulsion term. Orbital-dependent methods often cure most of the SIE problem, but they can be expensive to run. The ‘gap problem’ in DFT often stems from treating the KS HOMO-LUMO gap as the fundamental gap, but the difference in the HOMO and LUMO of the KS system is not the same as the difference between the ionization potential and the electron affinity[37]. Ad hoc methods are often used to correct DFT gaps, but these methods require expensive additional calculations, empirical knowledge of your system, or empirical tuning. However, it has recently been shown that some classes of self-interaction error are really just errors due to poor potentials leading to poorer densities [38, 39]. Such errors are removed by using more accurate densities (Fig. 4).

Range-separated hybrids: Range-separated hybrids[41] improve fundamental gaps calculated via the DFT HOMO-LUMO gap[42]. Screened range-separated hybrids can even achieve gap renormalization present when moving between gas-phase molecules and molecular crystals[43]. The basic range-separated hybrid scheme

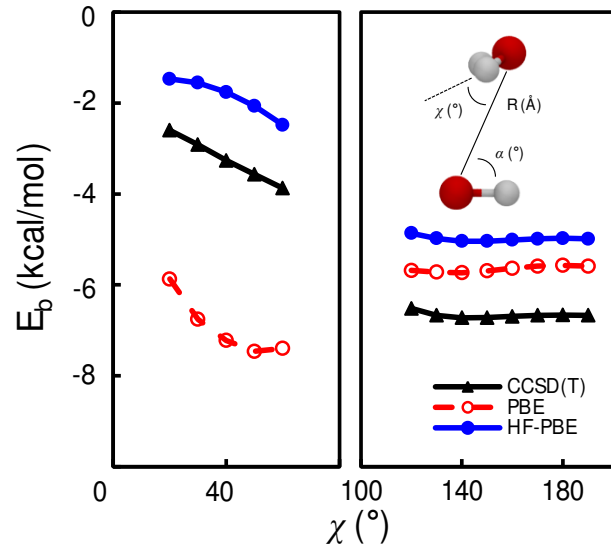


FIG. 4. When a DFT calculation is *abnormally* sensitive to the potential, the density can go bad. Usually, DFT approximate densities are better than HF[40], as in Fig 1. Here, self-consistent PBE results for $OH - H_2O$ interactions yield the wrong geometry, but PBE on HF-densities fixes this[39].

separates the troublesome Coulomb interaction into long-range and short-range pieces. The screened version enforces exact conditions to determine where this separation occurs and incorporates the dielectric constant as an adaptive parameter. This technique takes into account increased screening as molecules form solids, resulting in reduced gaps critical for calculations geared toward applications in molecular electronics.

Weak Interactions: Another of DFT’s classic failings is its poor treatment of weak interactions[44, 45]. Induced dipoles and the resulting dispersion interactions are not captured by the most popular approximations of Eq. 4. This prevents accurate modeling of the vast majority of biological systems, as well as a wide range of other phenomena, such as surface adsorption and molecular crystal packing. GGAs and hybrids are unable to model the long-range correlations occurring between fluctuations induced in the density. The non-empirical approach based on the work of Langreth and Lundqvist[46–49] and the empirical DFT-D of Grimme[50, 51] have dominated the advances in this area, along with the more recent, less empirical approach of Tkatchenko and Scheffler[52, 53].

IV. REDUCING COST: IS LESS MORE?

No matter how much progress is made in improving algorithms to reduce the computational cost of DFT calculations, there will always be larger systems of interest, and even the fastest calculations become prohibitively expensive. The most glaring example is molecular dy-

namics (MD) simulation in biochemistry. With classical force fields, these can be run for nano- to milli-seconds, with a million atoms, with relative ease. But when bonds break, a quantum treatment is needed, and the first versions of these were recognized in last year’s Nobel prize in chemistry[54–56]. These days, many people run Car-Parrinello MD[57, 58], with DFT calculations inside their MD, but this reduces tractable system sizes to a few hundred atoms.

Because of this, there remains a great deal of interest in finding clever ways to keep as much accuracy as needed while simplifying computational steps. One method for doing so involves circumventing the orbital-dependent KS step of traditional DFT calculations. Alternatively, one can save time by only doing those costly steps (or even more expensive procedures) on a system’s most important pieces, while leaving the rest to be calculated using a less intensive method. The key to both approaches is to achieve efficiency without sacrificing precious accuracy.

Removing the orbitals: Orbital-free methods[6, 59–63] like TF reduce computational costs, but are often not accurate enough to compete with KS DFT calculations. Current methods search for a similar solution, by working on non-interacting kinetic energy functionals that allow continued use of existing XC functionals[64]. (An intriguing alternative is to use the potential as the basic variable [65, 66] – see Secs. VI and VII.)

Embedding: Partitioning and embedding are similar procedures, in which calculations on isolated pieces of a molecule are used to gain understanding of the molecule as a whole[67]. One might want to separate out molecular regions to look more closely at pieces of high interest or to find a better way to approximate the overall energy with density functionals. Parsing a molecule into chunks can also allow for entirely new computational approaches not possible when dealing with the molecule as a whole.

Partition DFT[68] is an exact embedding method based on density partitioning[69, 70]. Because it uses ensemble density functionals[71, 72], it can handle non-integer electron numbers and spins[73]. Energy of the fragments is minimized using effective potentials consisting of a fragment’s potential and a global partition potential that maintains the correct total density. This breakdown into fragment and partition energies allows approximations good for localized systems to be used alongside those better for the extended effects associated with the partition potential.

While partition DFT uses DFT methods to break up the system, projector-based wavefunction-theory-in-DFT embedding techniques combine wavefunction and DFT methods[74, 75]. This multiscale approach leverages the increased accuracy of some wavefunction methods for some bonds, where high accuracy is vital, without extending this computational cost to the entire system. Current progress in this field has been toward the reduction of the errors introduced by the mismatch of methods between subsystems. This type of embedding has been

recently applied to heterolytic bond cleavage and conjugated systems[76]. Density matrix embedding theory on lattices[77] and its extension to full quantum mechanical chemical systems[78] use ideas from the density matrix renormalization group (DMRG)[79, 80], a blazingly fast way to exactly solve low-dimensional quantum mechanics problems. This shifts the interactions between fragments to a quantum bath instead of dealing with them through a partition potential.

V. WHAT IS THE UNDERLYING THEORY BEHIND DFT APPROXIMATIONS?

Given the Pandora’s box of approximate functionals, many found by fitting energies of systems, most users imagine DFT as an empirical hodgepodge. Ultimately, if we end up with a different functional for every system, we will have entirely defeated the idea of first-principles calculations. However, prior to the mid-90s, many decades of theory were developed to better understand the local approximation and how to improve on it[44]. Here we summarize the most relevant points.

The joint probability of finding one electron in a little chunk of space around point A and another in some other chunk of space around point B is called the pair probability density. The exact quantum repulsion among electrons is then

$$ElecRep = \frac{1}{2} \int dA \int dB \frac{P(A, B)}{|\mathbf{r}_A - \mathbf{r}_B|}. \quad (6)$$

But we can also write

$$P(A, B) = \rho(A) \rho_{\text{cond}}(A, B). \quad (7)$$

where $\rho(A)$ is the density at \mathbf{r}_A and $\rho_{\text{cond}}(A, B)$ is the probability of finding the second electron at B , *given* that there’s one at A . (If you ignore the electron at A , this is just $\rho(B)$, and Eq. 6 gives the Coulomb repulsion in Eqs. 2 and 3). We write this conditional probability as

$$\rho_{\text{cond}}(A, B) = \rho(B) + \rho_{\text{xc}}(A, B). \quad (8)$$

where $\rho_{\text{xc}}(A, B)$ is called the hole around A . It is mostly negative and represents a missing electron (it integrates to -1), since the conditional probability integrates to $N - 1$. With a little math trick (called the adiabatic connection[81, 82]), we can fold the kinetic correlation into the hole so that

$$XC = \frac{1}{2} \int dA \int dB \frac{\rho(A) \rho_{\text{xc}}(A, B)}{|\mathbf{r}_A - \mathbf{r}_B|}. \quad (9)$$

Because the XC hole tends to follow an electron around, i.e., be centered on A as in Fig. 5, its shape is roughly a simple function of $\rho(A)$. If one approximates the hole by that of a uniform gas of density $\rho(A)$, Eq. 9 above yields the LDA for the XC energy. So the LDA approximation for XC can be thought of as approximating the hole by that of a uniform gas[44, 83].

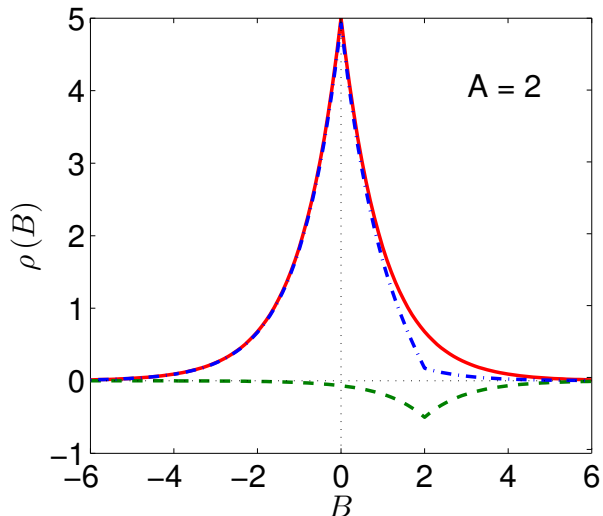


FIG. 5. Cartoon of a one-dimensional 10-electron density (solid red), the conditional density (dot-dashed blue) given an electron at $A = 2$, and its hole density (dashed green).

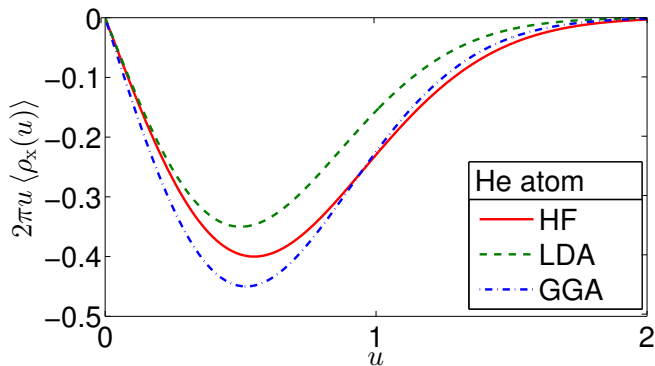


FIG. 6. Representation of system-averaged radial exchange holes for the helium atom[84], weighted by the Coulomb repulsion, so that the area equals the X energy. The LDA hole (dashed green) is not deep enough, reflecting the LDA underestimate of the magnitude of the X energy. The GGA hole (dot-dashed blue) is substantially better, but a little too deep.

But while the XC is roughly approximated by LDA, the energy density at each point in a system is not, especially in systems of low symmetry. But from Eq. 9, the energy depends only on the *average* of the XC hole over the system, and Fig. 6 shows such a system-averaged hole for the He atom. (Integrate over A and the angular parts of B in Eq. (9).) The LDA hole is not deep enough, and neither is the LDA energy. This is the effect that leads to LDA overbinding of molecules.

GGA Made Briefer: The underlying idea behind the Perdew series of GGAs was to improve on the LDA hole[85]. Adding gradient corrections to the hole violates certain sum rules (negativity of the X hole and integration to -1, and integration to 0 for the correlation hole), so the real-space cutoff procedure was designed to restore

these conditions. This is an effective resummation of the gradient expansion, producing the numerical GGA. The popular functional PBE was derived from imposing exact conditions on a simple form[28, 29], but should be believed because it mimics the numerical GGA. In Fig. 6, we show how the GGA hole roughly improves on LDA, reducing typical energy errors by a factor of three.

GGAs don't only show how important good hole models can be. They also demonstrate that good approximations can satisfy different exact conditions, so picking which to satisfy is non-trivial. For instance, B88[23], PW91[86, 87], and PBE[28, 29] give similar values for exchange energy when densities do not get too small or vary too quickly. However, once they do, each behaves very differently. Each approximation was sculpted to satisfy different exact conditions in this limit. Becke decided a good energy density for exponential electronic densities was important. Perdew first thought that a particular scaling behavior was important[88], then that satisfying a certain bound was better[28]. Without a systematic way to improve our approximations, these difficult choices guide our progress. But starting from a model for the XC hole is an excellent idea, as such a model can be checked against the exact XC hole[89].

XDM: A recent, parameter-free approach to capturing dispersion is the exchange-hole dipole moment (XDM) method[90–93], where perturbation theory yields a multipole-multipole interaction, and quantum effects are included through the dipole moment of the electron with its exchange hole. Using these in concert with atomic polarizabilities and dipole moments generates atomic pair dispersion coefficients that are within 4% of reference C_6 values[94]. Such a model has the advantage over the more popular methods of Sec. III because its assertions about the hole can be checked.

RPA and other methods: Originally put forth in the 1950s as a method for the uniform electron gas, the random phase approximation (RPA) can be viewed as a simplified wavefunction method or a nonlocal density functional approach that uses both occupied and unoccupied KS states to approximate the correlation energy. RPA correlation performs extremely well for noncovalent, weak interactions between molecules and yields the correct dissociation limit of H_2 [95], two of the major failures of traditional DFT approximations[96].

Though computational expense once hindered its wide use, resolution-of-identity implementations[97, 98] have improved its efficiency, making RPA accessible to researchers interested in large molecular systems. RPA gives good dissociation energy for catalysts involving the breaking of transition-metal-ligand and carbon-carbon bonds in a system of over 100 atoms[99]. Though RPA handles medium- and long-range interactions very well, its trouble with short-range correlations invites development of methods that go ‘beyond RPA.’ RPA used in quantum chemistry usually describes only the particle-hole channel of the correlation, but another recent approach to RPA is the particle-particle RPA

(pp-RPA)[100]. pp-RPA is missing some correlation, which causes errors in total energies of atoms and small molecules. This nearly cancels out in reaction energy calculations and yields fairly accurate binding energies[101].

RPA and variations on it will likely lead to methods that work for both molecules and solids, and their computational cost will be driven down by algorithmic development. However, RPA is likely to remain substantially more expensive than a GGA calculation for the indefinite future. While it may rise to fill an important niche in quantum chemistry, producing comparably accurate energetics to modern functionals without any empiricism, such methods will not *replace* DFT as the first run for many calculations. Moreover, as with almost all ‘better’ methods than DFT, there appears to be no way to build in the good performance of older DFT approximations.

VI. IS THERE A SYSTEMATIC APPROACH TO FUNCTIONAL APPROXIMATION?

A huge intellectual gap in DFT development has been in the theory behind the approximations. This, as detailed above, has allowed the rise of empirical energy fitting. Even the most appealing non-empirical development seems to rely on picking and choosing which exact conditions the approximation should satisfy. Lately, even Perdew has resorted to one or two parameters in the style of Becke[102, 103], in order to construct a meta-GGA. Furthermore, up until the mid 1990s, many good approximations were developed as approximations to the XC hole, which could then be tested and checked for simple systems.

However, in fact, there *is* a rigorous way to develop density functional approximations. Its mathematical foundations were laid down 40 years ago by Lieb and Simon[104–106]. They showed the fractional error in the energy in any TF calculation vanishes as $Z \rightarrow \infty$, keeping $N = Z$. Their original proof is for atoms, but applies to any molecule or solid, once the nuclear positions are scaled by $Z^{1/3}$ also. Their innocuous statement is in fact very profound. This very complicated many-body quantum problem, in the limit of *large* numbers of electrons, has an almost trivial (approximate) solution. And although the world finds TF theory too inaccurate to be useful, and performs KS calculations instead, the equivalent statement (not proven with rigor) is that the fractional error in the LDA for XC vanishes as $Z \rightarrow \infty$. XC, like politics, is entirely local in this limit.

These statements explain many of the phenomena we see in modern DFT:

- (i) LDA is *not* just an approximation that applies for uniform or slowly varying systems, but is instead a universal limit of *all* electronic systems.
- (ii) LDA is the leading term in an asymptotic expansion in powers of \hbar , i.e., semiclassical. Such expansions are notoriously difficult to deal with mathematically.
- (iii) The way in which LDA yields an ever smaller error

as Z grows is very subtle. The leading corrections are of several origins. Often the dominant error is a lack of spatial quantum oscillations in the XC hole. However, as Z grows, these oscillations get faster, and so their net effect on the XC energy becomes smaller. Thus, even as Z grows, LDA should not yield accurate energy densities everywhere in a system (and its potential is even worse, as in Fig. 1), but the integrated XC energy will become ever more accurate.

(iv) The basic idea of the GGA as the leading correction to LDA makes sense. The leading corrections to the LDA hole should exist as very sophisticated functionals of the potential, but whose energetic effects can be captured by simple approximations using the density gradient. This yields improved net energetics, but energy densities might look even worse, especially in regions of high gradients, such as atomic cores.

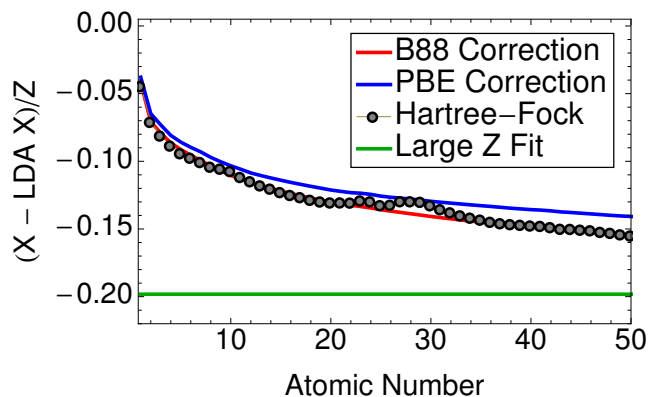


FIG. 7. The non-local exchange energy (exchange minus LDA X) per electron of atoms with atomic number Z (compare with Fig. 3). The PBE functional tends to the theoretical limit ($Z \rightarrow \infty$) (horizontal green line), but B88 is more accurate for $Z < 50$ because of fitting[107].

Next, we continue the allegory from Sec. II. To do so, we subtract the LDA X energy from our accurate ones, so we can see the next correction, and plot this, per electron, in Fig. 7. Now, a bright young chemist has heard about the GGA, cooks up an intuitive correction to LDA, and fits one parameter to the Nobel gas values. Later, some physicists derive a different GGA, which happens to also give the correct value. Later still, a derivation of the correction for large Z is given, which can be used to determine the parameter (and turns out to match the empirical value within 10%). The only difference from the original allegory is that this is all true. The chemist was Axel Becke; his fitted functional is B88[23]. The derived functional is PBE[28], and the derivation of the parameter in B88 is given in Ref. [107].

This true story both validates Becke’s original procedure and the semiclassical approach to density functional approximation. Note that even the correction is evaluated on the TF density to find the limiting behavior. The PBE exchange functional also yields the leading the

correction to the exchange energy of atoms. By throwing this away and restoring the (different) gradient expansion for slowly-varying gases, PBEsol was created[108].

Semiclassical approximations: New approximations driven by semiclassical research can be divided into density approaches and potential approaches. In the density camp, we find innovations like the approximations by Armiento and Mattsson[109–111], which incorporate surface conditions through their semiclassical approach. In the potential functional camp, we find highly accurate approximations to the density, which automatically generate approximations to non-interacting kinetic energies[65, 66, 112]. Since these approaches use potential functionals, they are orbital-free and incredibly efficient, but only apply in one dimension (see also Sec. VII). Current research is focused on extension to three dimensions, semiclassical approximations in the presence of classical turning points, as well as semiclassical approximations to exchange and correlation energies.

VII. WARM DENSE MATTER: A HOT NEW AREA?

Though we do not live at icy absolute zero, most chemistry and physics happens at low enough temperatures that electrons are effectively in their ground state. Most researchers pretend to be at zero temperature for their DFT work with impunity. But some people, either those working at high enough temperatures and pressures or those interested in low-energy transitions, can't ignore thermal effects. Those of us caught up in these warmer pursuits must tease out where temperature matters for our quantum mechanical work.

Mermin proved a finite-temperature version of the Hohenberg-Kohn theorem in 1965[113], and the finite-temperature LDA was shown in the original KS paper[21]. However, many people continue to rely on the zero-temperature approximations, though they populate states at higher energy levels using finite-temperature weightings. Better understanding and modeling of the finite-temperature XC hole could lead to improvement in some of the finer details of these calculations, like optical and electronic properties[114].

WDM and MD: One area that has seen great recent progress with DFT is the study of warm dense matter (WDM)[115, 116]. WDM is intermediate to solids and plasmas, inhabiting a world where both quantum and classical effects are important. It is found deep within planetary interiors, during shock physics experiments, and on the path to ignition of inertial confinement fusion. Lately, use of DFT MD has been a boon to researchers working to simulate these complicated materials[117–121]. Most of these calculations are performed using KS orbitals with thermal occupations, ignoring any temperature dependence of XC, in hopes that the kinetic and Coulomb energies will capture most of the thermal effects. Agreement with experiment has been excellent,

though there is great interest in seeing if temperature-dependent XC approximations affects these results.

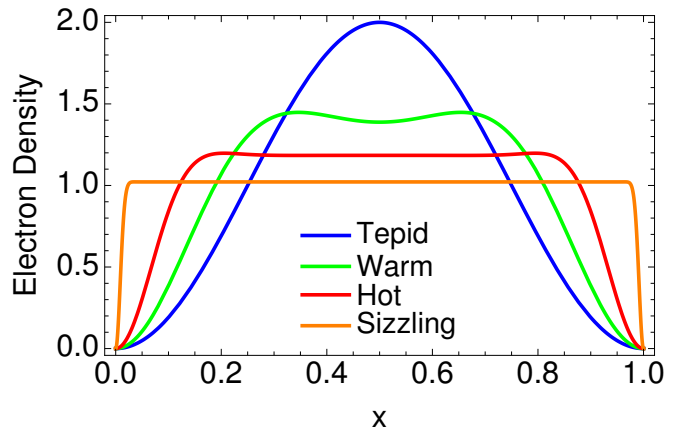


FIG. 8. The density of a single electron in a flat box spreads toward the infinite walls as temperatures rise.

Exact conditions: Exact conditions have been derived[114, 122–124] for finite-temperature systems that seem very similar to their ground-state counterparts. However, a major difference in thermal systems is that when one squeezes or compresses the length scale of the system, one sees an accompanying scaling of the temperature. This is further reflected in the thermal adiabatic connection, which connects the non-interacting KS system to the interacting system through scaling of the electron-electron interaction. At zero temperature, this allows us to write the XC energy in terms of the potential alone, as long as it is accompanied by appropriate squeezing or stretching of the system's length scale (see Sec. V). With the temperature-coordinate scaling present in thermal ensembles, the thermal adiabatic connection requires not only length scaling, but also the correct temperature scaling.

OF Methods: Orbital-free methods, discussed in Sec. IV, are of particular interest in the WDM community. Solving the KS equations with many thermally populated orbitals is repeated over and over in DFT MD, leading to prohibitive cost as temperatures rise. The focus on free energies for thermal ensembles has led to two different approaches to orbital-free approximations. One approach uses two separate forms for kinetic and entropic contributions[124]. Following this path, one can either make approximations empirically[126] or non-empirically[127]. Another approach enforces a particular type of response in the uniform gas limit[128]. If one wishes to approximate the kentropy[114] as a whole, one can use temperature-dependent potential functional theory to generate highly accurate approximations from approximate densities generated semiclassically or stochastically[125, 129]. Fig. 9 shows the accuracy of a semiclassical density approximation, which captures the quantum oscillations missed by Thomas-Fermi and still present as temperatures rise.

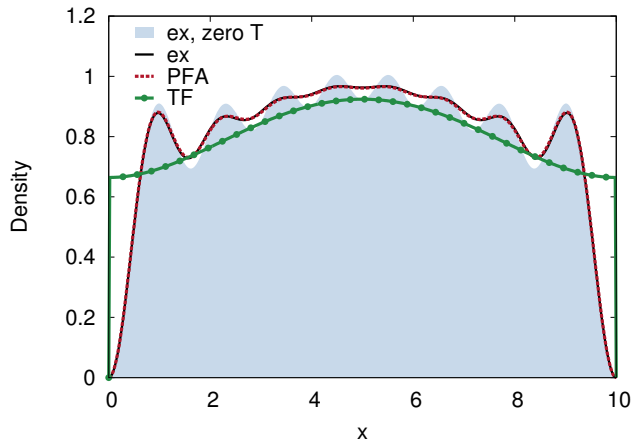


FIG. 9. Eight electrons in the potential $-2\sin^2(\pi x/10)$ in a 1d box. At zero temperature (gray), the density exhibits sharp quantum oscillations, which wash out as the temperature increases (black). This effect is much weaker near the edges. TF is used in many warm simulations, but (green) misses all oscillations, vital for accurate chemical effects. The orbital-free, finite-temperature potential functional approximation of Ref. [125] is almost exact here (red).

VIII. WHAT CAN WE GUESS ABOUT THE FUTURE?

The future of DFT remains remarkably bright. As Fig. 2 shows, the number of applications continues to grow exponentially, with three times as much activity than previously realized (Fig. 1 of [7]). While empiricism has generated far too many possible alternatives, the standard well-derived approximations continue to dominate.

To avoid losing insight, it is important to further develop the systematic path to approximations, which eschews all empiricism and expands the functional in powers of \hbar , Planck’s constant. This will ultimately tell us what we can and cannot do with local-type approximations. There is huge room for development in this area, and any progress could impact all those applications.

Meanwhile, new areas have been (e.g. weak interactions) or are being developed (warm dense matter). New methods, such as using Bayesian statistics for error analysis[130] or machine learning for finding functionals[63, 131], are coming on line. Such methods will not suffer the limitations of local approximations, and should be applicable to strongly correlated electronic systems, an arena where many of our present approximations fail. We have little doubt that DFT will continue to thrive for decades to come.

IX. ACKNOWLEDGMENTS

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- [1] S. D. Wong, M. Srnec, M. L. Matthews, L. V. Liu, Y. Kwak, K. Park, C. B. Bell III, E. E. Alp, J. Zhao, Y. Yoda, S. Kitao, M. Seto, C. Krebs, J. M. Bollinger, and E. I. Solomon, “Elucidation of the Fe(IV)=O intermediate in the catalytic cycle of the halogenase SyrB2,” *Nature*, vol. 499, pp. 320–323, 07 2013.
 - [2] M. D. Knudson, M. P. Desjarlais, R. W. Lemke, T. R. Mattsson, M. French, N. Nettelmann, and R. Redmer, “Probing the interiors of the ice giants: Shock compression of water to 700 gpa and 3.8 g/cm³,” *Phys. Rev. Lett.*, vol. 108, p. 091102, Feb 2012.
 - [3] D. Rappoport, N. R. M. Crawford, F. Furche, and K. Burke, “Which functional should I choose?,” in *Computational Inorganic and Bioinorganic Chemistry* (E. Solomon, R. King, and R. Scott, eds.), Wiley, John & Sons, Inc., 2009.
 - [4] K. Burke and L. O. Wagner, “D,” *Int. J. Quant. Chem.*
 - [5] R. G. Parr and W. Yang, *Density Functional Theory of Atoms and Molecules*. Oxford University Press, 1989.
 - [6] R. M. Dreizler and E. K. U. Gross, *Density Functional Theory: An Approach to the Quantum Many-Body Problem*. Berlin: Springer-Verlag, 1990.
 - [7] K. Burke, “Perspective on density functional theory,” *J. Chem. Phys.*, vol. 136, 2012.
 - [8] R. O. Jones, “Density functional theory: A personal view,” in *Strongly Correlated Systems* (A. Avella and F. Mancini, eds.), vol. 171 of *Springer Series in Solid-State Sciences*, pp. 1–28, Springer Berlin Heidelberg, 2012.
 - [9] A. Zangwill, “The education of Walter Kohn and the creation of density functional theory,” *Archive for History of Exact Sciences*, pp. 1–74, 2014.
 - [10] E. Schrödinger, “An undulatory theory of the mechanics of atoms and molecules,” *Phys. Rev.*, vol. 28, pp. 1049–1070, Dec 1926.
 - [11] L. H. Thomas, “The calculation of atomic fields,” *Math. Proc. Camb. Phil. Soc.*, vol. 23, no. 05, pp. 542–548, 1927.
 - [12] E. Fermi *Rend. Acc. Naz. Lincei*, vol. 6, 1927.
 - [13] E. Fermi, “Eine statistische Methode zur Bestimmung einiger Eigenschaften des Atoms und ihre Anwendung auf die Theorie des periodischen Systems der Elemente

- (A statistical method for the determination of some atomic properties and the application of this method to the theory of the periodic system of elements),” *Zeitschrift für Physik A Hadrons and Nuclei*, vol. 48, pp. 73–79, 1928.
- [14] J. C. Slater, “A Simplification of the Hartree-Fock Method,” *Phys. Rev.*, vol. 81, pp. 385–390, Feb 1951.
- [15] V. Fock, “Näherungsmethode zur lösung des quantenmechanischen mehrkörperproblems,” *Z. Phys.*, vol. 61, pp. 126–148, 1930.
- [16] D. R. Hartree and W. Hartree, “Self-consistent field, with exchange, for beryllium,” *Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences*, vol. 150, no. 869, pp. 9–33, 1935.
- [17] P. Hohenberg and W. Kohn, “Inhomogeneous electron gas,” *Phys. Rev.*, vol. 136, pp. B864–B871, Nov 1964.
- [18] E. Teller, “On the stability of molecules in the Thomas-Fermi theory,” *Rev. Mod. Phys.*, vol. 34, pp. 627–631, Oct 1962.
- [19] S. Kurth and J. P. Perdew, “Role of the exchange-correlation energy: Nature’s glue,” *International Journal of Quantum Chemistry*, vol. 77, no. 5, pp. 814–818, 2000.
- [20] R. J. Bartlett and M. Musial, “Coupled-cluster theory in quantum chemistry,” *Rev. Mod. Phys.*, vol. 79, pp. 291–352, Feb 2007.
- [21] W. Kohn and L. J. Sham, “Self-consistent equations including exchange and correlation effects,” *Phys. Rev.*, vol. 140, pp. A1133–A1138, Nov 1965.
- [22] J. Perdew, “Density functional approximation for the correlation energy of the inhomogeneous gas,” *Phys. Rev. B*, vol. 33, p. 8822, 1986.
- [23] A. D. Becke, “Density-functional exchange-energy approximation with correct asymptotic behavior,” *Phys. Rev. A*, vol. 38, pp. 3098–3100, Sep 1988.
- [24] A. D. Becke, “Density-functional thermochemistry. III. The role of exact exchange,” *The Journal of Chemical Physics*, vol. 98, no. 7, pp. 5648–5652, 1993.
- [25] J. P. Perdew and K. Schmidt in *Density Functional Theory and Its Applications to Materials* (V. E. V. Doren, K. V. Alsenoy, and P. Geerlings, eds.), Melville, NY: American Institute of Physics, 2001.
- [26] J. P. Perdew and S. Kurth, *Density Functionals for Non-relativistic Coulomb Systems in the New Century*, pp. 1–55. Berlin / Heidelberg: Springer, 2003.
- [27] J. Tao, J. P. Perdew, V. N. Staroverov, and G. E. Scuseria, “Climbing the density functional ladder: Nonempirical meta-generalized gradient approximation designed for molecules and solids,” *Phys. Rev. Lett.*, vol. 91, p. 146401, Sep 2003.
- [28] J. P. Perdew, K. Burke, and M. Ernzerhof, “Generalized gradient approximation made simple,” *Phys. Rev. Lett.*, vol. 77, pp. 3865–3868, Oct 1996. *ibid.* **78**, 1396(E) (1997).
- [29] K. B. J.P. Perdew and M. Ernzerhof, “Perdew, Burke, and Ernzerhof reply,” *Phys. Rev. Lett.*, vol. 80, p. 891, 1998.
- [30] C. Lee, W. Yang, and R. G. Parr, “Development of the Colle-Salvetti correlation-energy formula into a functional of the electron density,” *Phys. Rev. B*, vol. 37, pp. 785–789, Jan 1988.
- [31] C. Fischer, “A multi-configuration Hartree-Fock program,” *Computer Physics Communications*, vol. 1, no. 3, pp. 151 – 166, 1969.
- [32] C. Fischer, *Hartree-Fock method for atoms. A numerical approach*. Wiley, Jan 1977.
- [33] P. A. M. Dirac, “Note on exchange phenomena in the Thomas atom,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 26, no. 03, pp. 376–385, 1930.
- [34] J. Schwinger, “Thomas-Fermi model: The second correction,” *Phys. Rev. A*, vol. 24, pp. 2353–2361, Nov 1981.
- [35] N. C. Handy and A. J. Cohen, “Left-right correlation energy,” *Molecular Physics*, vol. 99, no. 5, pp. 403–412, 2001.
- [36] In fact, sadly, the young chemist is unable to find a permanent position, and she ends up selling parametrized functionals for food on the streets. On the other hand, the physicists all celebrate their triumph over empiricism with a voyage on a brand new ship, which has been designed with materials whose properties have been calculated using DFT. Because the local approximation, as given above, underestimates the magnitude of the exchange energy, the brittle transition temperature is overestimated. When the new ship sails through icy waters, its hull is weakened and damaged by an iceberg, so all of them drown. (The interested reader may find more information on the ductile-to-brittle transition in Ref. [132] and other works by Kaxiras.)
- [37] C. Fiolhais, F. Nogueira, and M. Marques, *A Primer in Density Functional Theory*. New York: Springer-Verlag, 2003.
- [38] M.-C. Kim, E. Sim, and K. Burke, “Understanding and reducing errors in density functional calculations,” *Phys. Rev. Lett.*, vol. 111, p. 073003, Aug 2013.
- [39] M.-C. Kim, E. Sim, and K. Burke, “Ions in solution: Density corrected density functional theory (DC-DFT),” *The Journal of Chemical Physics*, vol. 140, no. 18, p. 18A528, 2014.
- [40] E. J. Baerends, O. V. Gritsenko, and R. van Meer, “The Kohn-Sham gap, the fundamental gap and the optical gap: the physical meaning of occupied and virtual Kohn-Sham orbital energies,” *Phys. Chem. Chem. Phys.*, vol. 15, pp. 16408–16425, 2013.
- [41] J. Heyd, G. E. Scuseria, and M. Ernzerhof, “Hybrid functionals based on a screened coulomb potential,” *The Journal of Chemical Physics*, vol. 118, no. 18, pp. 8207–8215, 2003.
- [42] L. Kronik, T. Stein, S. Refaely-Abramson, and R. Baer, “Excitation gaps of finite-sized systems from optimally tuned range-separated hybrid functionals,” *Journal of Chemical Theory and Computation*, vol. 8, no. 5, pp. 1515–1531, 2012.
- [43] S. Refaely-Abramson, S. Sharifzadeh, M. Jain, R. Baer, J. B. Neaton, and L. Kronik, “Gap renormalization of molecular crystals from density-functional theory,” *Phys. Rev. B*, vol. 88, p. 081204, Aug 2013.
- [44] R. Jones and O. Gunnarsson, “The density functional formalism, its applications and prospects,” *Rev. Mod. Phys.*, vol. 61, p. 689, 1989.
- [45] J. D. E.K.U. Gross and M. Petersilka, “Density functional theory of time-dependent phenomena,” *Topics in Current Chemistry*, vol. 181, p. 81, 1996.
- [46] Y. Andersson, D. Langreth, and B. Lunqvist, “van der Waals Interactions in Density-Functional Theory,” *Phys. Rev. Lett.*, vol. 76, p. 102, 1996.

- [47] M. Dion, H. Rydberg, E. Schröder, D. C. Langreth, and B. I. Lundqvist, “Van der Waals density functional for general geometries,” *Phys. Rev. Lett.*, vol. 92, p. 246401, Jun 2004.
- [48] J. M. Soler, E. Artacho, J. D. Gale, A. García, J. Junquera, P. Ordejón, and D. Sánchez-Portal, “The SIESTA method for ab initio order-N materials simulation,” *Journal of Physics: Condensed Matter*, vol. 14, no. 11, p. 2745, 2002.
- [49] K. Lee, A. K. Kelkkanen, K. Berland, S. Andersson, D. C. Langreth, E. Schröder, B. I. Lundqvist, and P. Hyldgaard, “Evaluation of a density functional with account of van der Waals forces using experimental data of H₂ physisorption on Cu(111),” *Phys. Rev. B*, vol. 84, p. 193408, Nov 2011.
- [50] S. Grimme, “Semiempirical GGA-type density functional constructed with a long-range dispersion correction,” *Journal of Computational Chemistry*, vol. 27, no. 15, pp. 1787–1799, 2006.
- [51] P. Jurecka, J. Sponer, J. Cerny, and P. Hobza, “Benchmark database of accurate (MP2 and CCSD(T) complete basis set limit) interaction energies of small model complexes, DNA base pairs, and amino acid pairs,” *Phys. Chem. Chem. Phys.*, vol. 8, pp. 1985–1993, 2006.
- [52] A. Tkatchenko and M. Scheffler, “Accurate Molecular Van Der Waals Interactions from Ground-State Electron Density and Free-Atom reference data,” *Phys. Rev. Lett.*, vol. 102, p. 073005, Feb 2009.
- [53] G.-X. Zhang, A. Tkatchenko, J. Paier, H. Appel, and M. Scheffler, “van der Waals Interactions in Ionic and Semiconductor Solids,” *Phys. Rev. Lett.*, vol. 107, p. 245501, Dec 2011.
- [54] A. Warshel and M. Levitt, “Theoretical studies of enzymic reactions: Dielectric, electrostatic and steric stabilization of the carbonium ion in the reaction of lysozyme,” *Journal of Molecular Biology*, vol. 103, no. 2, pp. 227 – 249, 1976.
- [55] M. Levitt, “The birth of computational structural biology,” *Nature Structural Biology*, vol. 8, pp. 392–393, 2001.
- [56] M. Karplus, “Spinach on the ceiling: A theoretical chemist’s return to biology,” *Annual Review of Biophysics and Biomolecular Structure*, vol. 35, no. 1, pp. 1–47, 2006.
- [57] R. Car and M. Parrinello, “Unified approach for molecular dynamics and density-functional theory,” *Phys. Rev. Lett.*, vol. 55, pp. 2471–2474, Nov 1985.
- [58] R. Iftimie, P. Minary, and M. E. Tuckerman, “Ab initio molecular dynamics: Concepts, recent developments, and future trends,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 102, no. 19, pp. 6654–6659, 2005.
- [59] Y. A. Wang and E. A. Carter, “Orbital-free kinetic-energy density functional theory,” in *Theoretical Methods in Condensed Phase Chemistry* (S. D. Schwartz, ed.), ch. 5, p. 117, Dordrecht: Kluwer, 2000.
- [60] V. V. Karasiev, R. S. Jones, S. B. Trickey, and F. E. Harris, “Properties of constraint-based single-point approximate kinetic energy functionals,” *Phys. Rev. B*, vol. 80, p. 245120, Dec 2009.
- [61] V. V. Karasiev, R. S. Jones, S. B. Trickey, and F. E. Harris, “Erratum: Properties of constraint-based single-point approximate kinetic energy functionals [Phys. Rev. B 80, 245120 (2009)],” *Phys. Rev. B*, vol. 87, p. 239903, Jun 2013.
- [62] V. Karasiev and S. Trickey, “Issues and challenges in orbital-free density functional calculations,” *Computer Physics Communications*, vol. 183, no. 12, pp. 2519 – 2527, 2012.
- [63] J. C. Snyder, M. Rupp, K. Hansen, K.-R. Mueller, and K. Burke, “Finding density functionals with machine learning,” *Phys. Rev. Lett.*, vol. 108, p. 253002, 2012.
- [64] W. Koch and M. C. Holthausen, *A Chemist’s Guide to Density Functional Theory*. Weinheim: Wiley-VCH, second ed., 2002.
- [65] A. Cangi, D. Lee, P. Elliott, K. Burke, and E. K. U. Gross, “Electronic structure via potential functional approximations,” *Phys. Rev. Lett.*, vol. 106, p. 236404, Jun 2011.
- [66] A. Cangi, E. K. U. Gross, and K. Burke, “Potential functionals versus density functionals,” *Phys. Rev. A*, vol. 88, no. 6, 2013.
- [67] H. Lin and D. Truhlar, “QM/MM: what have we learned, where are we, and where do we go from here?,” *Theoretical Chemistry Accounts*, vol. 117, no. 2, pp. 185–199, 2007.
- [68] P. Elliott, K. Burke, M. H. Cohen, and A. Wasserman, “Partition density-functional theory,” *Phys. Rev. A*, vol. 82, p. 024501, Aug 2010.
- [69] M. Cohen and A. Wasserman, “On hardness and electronegativity equalization in chemical reactivity theory,” *Journal of Statistical Physics*, vol. 125, pp. 1121–1139, 2006. 10.1007/s10955-006-9031-0.
- [70] M. H. Cohen and A. Wasserman, “On the foundations of chemical reactivity theory,” *The Journal of Physical Chemistry A*, vol. 111, no. 11, pp. 2229–2242, 2007.
- [71] L. O. E.K.U. Gross and W. Kohn, “Density-functional theory for ensembles of fractionally occupied states. i. basic formalism,” *Phys. Rev. A*, vol. 37, p. 2809, 1988.
- [72] A. Pribram-Jones, Z.-H. Yang, J. R. Trail, K. Burke, R. J. Needs, and C. A. Ullrich, “Excitations and benchmark ensemble density functional theory for two electrons,” *J. Chem. Phys.*, vol. 140, p. 18A541, 2014.
- [73] R. Tang, J. Nafziger, and A. Wasserman, “Fragment occupations in partition density functional theory,” *Phys. Chem. Chem. Phys.*, vol. 14, pp. 7780–7786, 2012.
- [74] F. R. Manby, M. Stella, J. D. Goodpaster, and T. F. Miller, “A simple, exact density-functional-theory embedding scheme,” *Journal of Chemical Theory and Computation*, vol. 8, no. 8, pp. 2564–2568, 2012.
- [75] T. A. Barnes, J. D. Goodpaster, F. R. Manby, and T. F. Miller, “Accurate basis set truncation for wavefunction embedding,” *The Journal of Chemical Physics*, vol. 139, no. 2, 2013.
- [76] J. D. Goodpaster, T. A. Barnes, F. R. Manby, and T. F. Miller, “Accurate and systematically improvable density functional theory embedding for correlated wavefunctions,” *The Journal of Chemical Physics*, vol. 140, no. 18, 2014.
- [77] G. Knizia and G. K.-L. Chan, “Density matrix embedding: A simple alternative to dynamical mean-field theory,” *Phys. Rev. Lett.*, vol. 109, p. 186404, Nov 2012.
- [78] G. Knizia and G. K.-L. Chan, “Density matrix embedding: A strong-coupling quantum embedding theory,” *arXiv:1212.2679*, 2013.
- [79] S. R. White, “Density matrix formulation for quantum renormalization groups,” *Phys. Rev. Lett.*, vol. 69, pp. 2863–2866, Nov 1992.

- [80] S. R. White, "Density-matrix algorithms for quantum renormalization groups," *Phys. Rev. B*, vol. 48, pp. 10345–10356, Oct 1993.
- [81] D. Langreth and J. Perdew, "The exchange-correlation energy of a metallic surface," *Solid State Commun.*, vol. 17, p. 1425, 1975.
- [82] O. Gunnarsson and B. Lundqvist, "Exchange and correlation in atoms, molecules, and solids by the spin-density-functional formalism," *Phys. Rev. B*, vol. 13, p. 4274, 1976.
- [83] K. B. M. Ernzerhof and J. Perdew, "Long-range asymptotic behavior of ground-state wavefunctions," *J. Chem. Phys.*, vol. 105, p. 2798, 1996.
- [84] M. Ernzerhof and J. Perdew, "Generalized gradient approximation to the angle- and system-averaged exchange hole," *J. Chem. Phys.*, vol. 109, p. 3313, 1998.
- [85] A. Becke, "Density-functional thermochemistry .5. Systematic optimization of exchange-correlation functionals," *Journal Of Chemical Physics*, vol. 107, pp. 8554–8560, Nov 22 1997.
- [86] S. V. K. J. M. P. D. S. J.P. Perdew, J.A. Chevary and C. Fiolhais, "Atoms, molecules, solids, and surfaces: Applications of the generalized gradient approximation for exchange and correlation," *Phys. Rev. B*, vol. 46, p. 6671, 1992.
- [87] J. Perdew, J. Chevary, S. Vosko, K. Jackson, M. Pederson, D. Singh, and C. Fiolhais, "Atoms, molecules, solids, and surfaces - applications of the generalized gradient approximation for exchange and correlation (vol 46, pg 6671, 1992)," *Physical Review B*, vol. 48, p. 4978, Aug 15 1993.
- [88] K. Burke, J. P. Perdew, and Y. Wang, *Derivation of a generalized gradient approximation: The PW91 density functional*, p. 81. NY: Plenum, 1997.
- [89] A. C. Cancio and C. Y. Fong, "Scaling properties of exchange and correlation holes of the valence shell of second-row atoms," *Phys. Rev. A*, vol. 85, p. 042515, Apr 2012.
- [90] E. R. Johnson and A. D. Becke, "Van der waals interactions from the exchange hole dipole moment: Application to bio-organic benchmark systems," *Chemical Physics Letters*, vol. 432, no. 4 - 6, pp. 600 - 603, 2006.
- [91] A. D. Becke and E. R. Johnson, "A unified density-functional treatment of dynamical, nondynamical, and dispersion correlations," *The Journal of Chemical Physics*, vol. 127, no. 12, 2007.
- [92] A. D. Becke and E. R. Johnson, "Exchange-hole dipole moment and the dispersion interaction revisited," *The Journal of Chemical Physics*, vol. 127, no. 15, 2007.
- [93] A. Otero-de-la Roza, B. H. Cao, I. K. Price, J. E. Hein, and E. R. Johnson, "Predicting the relative solubilities of racemic and enantiopure crystals by density-functional theory," *Angewandte Chemie International Edition*, vol. 53, no. 30, pp. 7879–7882, 2014.
- [94] A. Becke and E. Johnson, "Exchange-hole dipole moment and the dispersion interaction: High-order dispersion coefficients," *J. Chem. Phys.*, vol. 124, p. 014104, 2006.
- [95] M. Fuchs, Y.-M. Niquet, X. Gonze, and K. Burke, "Describing static correlation in bond dissociation by Kohn-Sham density functional theory," *The Journal of Chemical Physics*, vol. 122, no. 9, p. 094116, 2005.
- [96] A. J. Cohen, P. Mori-Sánchez, and W. Yang, "Insights into current limitations of density functional theory," *Science*, vol. 321, no. 5890, pp. 792–794, 2008.
- [97] F. Furche, "Developing the random phase approximation into a practical post-Kohn-Sham correlation model," *The Journal of Chemical Physics*, vol. 129, no. 11, 2008.
- [98] H. Eshuis, J. Yarkony, and F. Furche, "Fast computation of molecular random phase approximation correlation energies using resolution of the identity and imaginary frequency integration," *The Journal of Chemical Physics*, vol. 132, no. 23, p. 234114, 2010.
- [99] H. Eshuis and F. Furche, "A parameter-free density functional that works for noncovalent interactions," *The Journal of Physical Chemistry Letters*, vol. 2, no. 9, pp. 983–989, 2011.
- [100] H. van Aggelen, Y. Yang, and W. Yang, "Exchange-correlation energy from pairing matrix fluctuation and the particle-particle random phase approximation," *The Journal of Chemical Physics*, vol. 140, no. 18, 2014.
- [101] D. Peng, S. N. Steinmann, H. van Aggelen, and W. Yang, "Equivalence of particle-particle random phase approximation correlation energy and ladder-coupled-cluster doubles," *The Journal of Chemical Physics*, vol. 139, no. 10, 2013.
- [102] A. Z. P. B. J.P. Perdew, S. Kurth, "Accurate density functional with correct formal properties: A step beyond the generalized gradient approximation," *Phys. Rev. Lett.*, vol. 82, p. 2544, 1999.
- [103] J. Sun, R. Haunschild, B. Xiao, I. W. Bulik, G. E. Scuseria, and J. P. Perdew, "Semilocal and hybrid meta-generalized gradient approximations based on the understanding of the kinetic-energy-density dependence," *The Journal of Chemical Physics*, vol. 138, no. 4, 2013.
- [104] E. Lieb and B. Simon, "Thomas-Fermi theory revisited," *Phys. Rev. Lett.*, vol. 31, p. 681, 1973.
- [105] E. H. Lieb and B. Simon, "The Thomas-Fermi theory of atoms, molecules and solids," *Advances in Mathematics*, vol. 23, no. 1, pp. 22 - 116, 1977.
- [106] E. H. Lieb, "Thomas-fermi and related theories of atoms and molecules," *Rev. Mod. Phys.*, vol. 53, pp. 603–641, Oct 1981.
- [107] P. Elliott and K. Burke, "Non-empirical derivation of the parameter in the B88 exchange functional," *Can. J. Chem. Ecol.*, vol. 87, pp. 1485–1491, Oct 2009.
- [108] M. Ernzerhof and G. E. Scuseria, "Assessment of the Perdew-Burke-Ernzerhof exchange-correlation functional," *J. Chem. Phys.*, vol. 110, p. 5029, 1999.
- [109] R. Armiento and A. Mattsson, "Functional designed to include surface effects in self-consistent density functional theory," *Phys. Rev. B*, vol. 72, p. 085108, 2005.
- [110] A. E. Mattsson and R. Armiento, "Implementing and testing the AM05 spin density functional," *Phys. Rev. B*, vol. 79, p. 155101, Apr 2009.
- [111] A. E. Mattsson and R. Armiento, "The subsystem functional scheme: The Armiento-Mattsson 2005 (AM05) functional and beyond," *International Journal of Quantum Chemistry*, vol. 110, no. 12, pp. 2274–2282, 2010.
- [112] A. Cangi, D. Lee, P. Elliott, and K. Burke, "Leading corrections to local approximations," *Phys. Rev. B*, vol. 81, p. 235128, Jun 2010.
- [113] N. D. Mermin, "Thermal properties of the inhomogeneous electron gas," *Phys. Rev.*, vol. 137, p. A: 1441, 1965.
- [114] A. Pribram-Jones, S. Pittalis, E. Gross, and K. Burke, "Thermal density functional theory in context," in

- Frontiers and Challenges in Warm Dense Matter* (F. Graziani, M. P. Desjarlais, R. Redmer, and S. B. Trickey, eds.), vol. 96 of *Lecture Notes in Computational Science and Engineering*, pp. 25–60, Springer International Publishing, 2014.
- [115] N. R. C. C. on High Energy Density Plasma Physics Plasma Science Committee, *Frontiers in High Energy Density Physics: The X-Games of Contemporary Science*. The National Academies Press, 2003.
- [116] F. Graziani, M. P. Desjarlais, R. Redmer, and S. B. Trickey, eds., *Frontiers and Challenges in Warm Dense Matter*, vol. 96 of *Lecture Notes in Computational Science and Engineering*. Springer International Publishing, 2014.
- [117] T. R. Mattsson and M. P. Desjarlais, “Phase diagram and electrical conductivity of high energy-density water from density functional theory,” *Phys. Rev. Lett.*, vol. 97, p. 017801, Jul 2006.
- [118] B. Holst, R. Redmer, and M. P. Desjarlais, “Thermophysical properties of warm dense hydrogen using quantum molecular dynamics simulations,” *Phys. Rev. B*, vol. 77, p. 184201, May 2008.
- [119] A. Kietzmann, R. Redmer, M. P. Desjarlais, and T. R. Mattsson, “Complex behavior of fluid lithium under extreme conditions,” *Phys. Rev. Lett.*, vol. 101, p. 070401, Aug 2008.
- [120] M. D. Knudson and M. P. Desjarlais, “Shock Compression of Quartz to 1.6 TPa: Redefining a Pressure Standard,” *Phys. Rev. Lett.*, vol. 103, p. 225501, Nov 2009.
- [121] S. Root, R. J. Magyar, J. H. Carpenter, D. L. Hanson, and T. R. Mattsson, “Shock Compression of a Fifth Period Element: Liquid Xenon to 840 GPa,” *Phys. Rev. Lett.*, vol. 105, p. 085501, Aug 2010.
- [122] H. Eschrig, “ $T > 0$ ensemble-state density functional theory via Legendre transform,” *Phys. Rev. B*, vol. 82, p. 205120, Nov 2010.
- [123] S. Pittalis, C. R. Proetto, A. Floris, A. Sanna, C. Bersier, K. Burke, and E. K. U. Gross, “Exact conditions in finite-temperature density-functional theory,” *Phys. Rev. Lett.*, vol. 107, p. 163001, Oct 2011.
- [124] J. W. Dufty and S. B. Trickey, “Scaling, bounds, and inequalities for the noninteracting density functionals at finite temperature,” *Phys. Rev. B*, vol. 84, p. 125118, Sep 2011.
- [125] A. Cangi and A. Pribram-Jones, 2014. in preparation.
- [126] V. V. Karasiev, T. Sjostrom, and S. B. Trickey, “Generalized-gradient-approximation noninteracting free-energy functionals for orbital-free density functional calculations,” *Phys. Rev. B*, vol. 86, p. 115101, Sep 2012.
- [127] V. V. Karasiev, D. Chakraborty, O. A. Shukruto, and S. B. Trickey, “Nonempirical generalized gradient approximation free-energy functional for orbital-free simulations,” *Phys. Rev. B*, vol. 88, p. 161108, Oct 2013.
- [128] T. Sjostrom and J. Daligault, “Nonlocal orbital-free noninteracting free-energy functional for warm dense matter,” *Phys. Rev. B*, vol. 88, p. 195103, Nov 2013.
- [129] R. Baer, D. Neuhauser, and E. Rabani, “Self-Averaging Stochastic Kohn-Sham Density-Functional Theory,” *Phys. Rev. Lett.*, vol. 111, p. 106402, Sep 2013.
- [130] A. J. Medford, J. Wellendorff, A. Vojvodic, F. Studt, F. Abild-Pedersen, K. W. Jacobsen, T. Bligaard, and J. K. Nørskov, “Assessing the reliability of calculated catalytic ammonia synthesis rates,” *Science*, vol. 345, no. 6193, pp. 197–200, 2014.
- [131] J. C. Snyder, M. Rupp, K. Hansen, L. Blooston, K.-R. Müller, and K. Burke, “Orbital-free bond breaking via machine learning,” *The Journal of Chemical Physics*, vol. 139, no. 22, 2013.
- [132] G. Schusteritsch and E. Kaxiras, “Sulfur-induced embrittlement of nickel: a first-principles study,” *Modeling and Simulation in Materials Science and Engineering*, vol. 20, p. 065007, 2012.